Higher-Dimensional Realizability for Intensional Type Theory

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BHK

- Evidence for a conjunction of propositions is a pair consisting of evidence for each of the conjuncts.
- Evidence for a disjunction of propositions is a pair consisting of an identifier for one of the disjuncts as well as evidence for the identified proposition.
- Evidence for an implication is a process/method/function that, when given evidence for the antecedent, produces evidence for the consequent.
- Evidence for an existential proposition is a pair consisting of an object (from the domain of discourse) and evidence that this object satisfies the body of the proposition.
- Evidence for a universal proposition is again a process/method/function that, when given any object (of the domain of discourse), produces evidence that this object satisfies the body of the proposition.

Number realizability

- $n \Vdash t = t'$ iff $t = t'$.
- $n \Vdash \varphi \wedge \psi$ iff $\pi_1(n) \Vdash \varphi \wedge \pi_2(n) \Vdash \psi$.
- $n \Vdash \varphi \vee \psi$ iff one of the following hold:
  - $\pi_1(n) = 0$ and $\pi_2(n) \Vdash \varphi$.
  - $\pi_1(n) = 1$ and $\pi_2(n) \Vdash \psi$.
- $n \Vdash \varphi \to \psi$ iff for all $m \Vdash \varphi$: $\{n\}(m)\downarrow$ and $\{n\}(m) \Vdash B$.
- $n \Vdash \exists x.\varphi(x)$ iff $\pi_2(n) \Vdash \varphi(\pi_1(n))$.
- $n \Vdash \forall x.\varphi(x)$ iff for all $m \in \mathbb{N}$: $\{n\}(m)\downarrow$ and $\{n\}(m) \Vdash \varphi(m)$.

Higher-dimensional BHK

The homotopy interpretation of type theory:

$$\begin{aligned} \text{types} &\rightsquigarrow \text{spaces} \\ \text{terms} &\rightsquigarrow \text{points in a space} \\ \text{identifications} &\rightsquigarrow \text{paths in a space} \\ \text{higher identifications} &\rightsquigarrow \text{higher paths, ie. homotopies} \end{aligned}$$

BHK reading:

- Evidence for an identification is a path.

Goal

Construct realizability models of ITT that formalise the higher-dimensional BHK interpretation.

Realizers themselves should carry higher-dimensional structure.

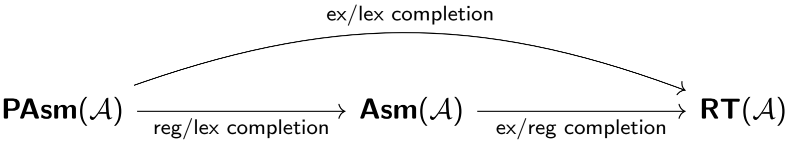
The notion of model that we'll use is that given by path categories. These have a nice theory of exact completions.

Categorical realizability

Realizability categories are often considered over PCAs (algebraic models of untyped computation), eg. Kleene's first algebra $\mathcal{K}_1$:

$$|\mathcal{K}_1| := \mathbb{N} \qquad n \cdot m := \{n\}(m)$$

$\mathbf{Eff} = \mathbf{RT}(\mathcal{K}_1)$: its internal logic extends number realizability.



Partitioned (set-based) assemblies

A partitioned assembly X over a PCA $\mathcal{A}$ is a set X together with a function

$$\|-\|_X : X \to \mathcal{A}$$

X is modest when $\|-\|_X$ is injective.

A morphism $X \to Y$ of partitioned assemblies is a function $f : X \to Y$ such that there exists $e \in \mathcal{A}$ such that for all $x \in X$:

$$e \cdot \|x\|_X = \|f(x)\|_Y$$

Related work

[Hofstra-Warren '13] equips syntax of 1-truncated ITT with a notion of realizability: the syntactic groupoid associated to the type theory generated by a graph has the same homotopy type as the free groupoid on this graph.

[Uemura '18] presents cubical assemblies model with impredicative and univalent universe that does not satisfy propositional resizing.

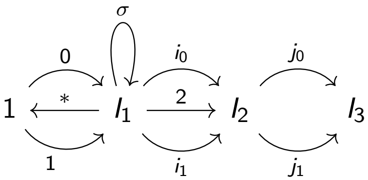
[van den Berg '18] exhibits **Eff** as homotopy category of a certain path category containing impredicative and univalent universe that does satisfy propositional resizing.

[Angiuli-Harper-Wilson '17] introduces computational higher type theory, which can be seen as a realizability construction over an untyped cubical λ-calculus.

Groupoidal realizability

Instead of underlying sets, we'll have underlying groupoids.

Instead of PCAs, we work with "realizer categories". A realizer category is a cartesian closed category $\mathbb{C}$ together with an interval $\mathbb{I}$ *qua* internal co-groupoid (à la [Warren '12]



Homotopies and fundamental groupoids

An interval $\mathbb{I} \in \mathbb{C}$ gives rise to a notion of homotopy in $\mathbb{C}$.

$$\begin{array}{ccccc} A \times 1 & \xrightarrow{A \times 0} & A \times I_1 & \xleftarrow{A \times 1} & A \times 1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \pi_1} \\ A & \xrightarrow[f]{} & B & \xleftarrow[g]{} & A \end{array}$$

$\mathbb{C}$ is elevated to a strict ω-category, where higher cells are homotopies.

We also get a fundamental groupoid 2-functor:

$$\Pi := (-)^{\mathbb{I}} : \mathbb{C} \to \mathbf{Gpd}$$

PGAsm$(\mathbb{C}, \mathbb{I})$

A partitioned groupoidal assembly X is a triple consisting of groupoid X, a "realizer type" $A \in \mathbb{C}$ and a functor

$$\| - \|_X : X \to \Pi A$$

X is modest when $\|-\|_X$ is fully faithful.

A morphism $X \to Y$ of partitioned groupoidal assemblies is a functor $F : X \to Y$ such that there exists $e : A \to B$ and a natural iso:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle \|-\|_X}\downarrow & \overset{\epsilon}{\nearrow} & \downarrow{\scriptstyle \|-\|_Y} \\ \Pi A & \xrightarrow[\Pi(e)]{} & \Pi B \end{array}$$

Some properties

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is weakly cartesian closed. If Y is modest then so is the weak exponential Y^X.

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ has an interval of its own, by matching up the interval from groupoids with the interval $\mathbb{I} \in \mathbb{C}$. This works because morphisms are allowed to be realized up to natural iso.

If $(\mathbb{C}, \mathbb{I})$ is finitely complete as a (2,1)-category, then so is $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$.

Finitely complete realizer categories

Every map $\phi : A \times I_1 \times I_1 \to B$ determines a square $\partial(\phi) \in \square\mathbb{C}(A, B)$:

$$\begin{array}{ccc} \phi_{00} & \xrightarrow{\phi_{I0}} & \phi_{10} \\ {\scriptstyle \phi_{0I}}\downarrow & & \downarrow{\scriptstyle \phi_{1I}} \\ \phi_{01} & \xrightarrow[\phi_{I1}]{} & \phi_{11} \end{array}$$

Denote by $\blacksquare\mathbb{C}(A, B)$ the double category whose squares are maps $A \times I_1 \times I_1 \to B$.

$$\partial : \blacksquare\mathbb{C}(A, B) \to \square\mathbb{C}(A, B)$$

Lemma (Warren '12)

If $(\mathbb{C}, \mathbb{I})$ is a finitely complete as a (2,1)-category then ∂ is an isomorphism of double categories.

As a path category

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is a path category with:

- Equivalences: equivalences internal to the (2,1)-category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$,
- Fibrations: morphisms whose underlying functor is a Grothendieck fibration.

A path object for X is given by the weak exponential $X^{\mathbb{I}}$.

IF $F : X \to Y$ is a fibration, then for any $q : y \to y' \in Y$:

$$q^* : X_y \to X_{y'}$$

is a morphism in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ realized by (id, ϵ), where

$$\epsilon_x := \|\overline{q}(x)\|_X$$

Dependent products

There is a subclass of fibrations given by the "modest fibrations".

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ {\scriptstyle y^* M}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle M} \\ 1 & \underset{y}{\longrightarrow} & Y \end{array}$$

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ has weak homotopy dependent products.

If $F : Y \to Z$ is a fibration and $M : X \to Y$ is a modest fibration, then the dependent product $\Pi_F(M) : \Pi_F X \to Z$ is a modest fibration.

Theorem (van den Berg-Moerdijk '18)

The exact completion of a path category with weak homotopy dependent products is a path category with homotopy dependent products.

Weak *un*typed theory (universal objects)

A category provides a typed notion of realizability (objects of the category are types).

We're interested in impredicative universes, but "untypedness" is essential for impredicativity [Birkedal '00, Robinson-Rosolini '01, Lietz-Streicher '02].

To get an untyped notion of realizability, we posit a universal object $U \in \mathbb{C}$: every object in $\mathbb{C}$ is a retract of U up to homotopy.

$$A \xrightarrow{s_A} U \xrightarrow{r_A} A \qquad \rho_A : r_A s_A \Rightarrow \mathrm{id}_A$$

So U is an up-to-homotopy model of the untyped λ-calculus.

Impredicative universes

There is an equivalence

$$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U) \simeq \mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$$

where $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U) \subseteq \mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ is the full subcategory spanned by the objects whose realizer type is U.

$\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ has a representation for modest fibrations: there is a modest fibration $\theta : \Theta \to \Lambda$ such that every modest fibration is equivalent to the pullback of θ along some morphism.

Representation for modest fibrations

$$\Lambda := \nabla \left(\widehat{\Pi U} \right)$$

Θ has:

- Objects: pairs (F, a), where $F \in \widehat{\Pi U}$ and $Fa \neq \emptyset$.
- Morphisms: a morphism $(\psi, \alpha) : (F, a) \to (G, b)$ is a natural iso ψ and a path $\alpha : a \to b$.

On objects, the characteristic map $\nu_M : X \to \Lambda$ of M is given by:

$$\nu_M(x)(a) := \left\{ \alpha : a \to a' \mid \exists y \in Y.\, My = x \wedge \|y\|_Y = a' \right\} / \sim$$
$$\nu_M(x)(\beta : a \to b)[\alpha] := \left[\alpha \circ \beta^{-1} \right]$$

A Grothendieck correspondence

An indexed partitioned groupoidal assembly $\mathbb{X}$ over $X \in \mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ is a functor

$$\mathbb{X} : X \to \mathbf{Gpd} \downarrow \Pi U$$

One can construct an indexed partitioned groupoidal assembly over X from a split fibration over X, and vice versa.

Future work

- Is the exact completion of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ a higher topos?
- Does the regular or exact completion of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ contain an impredicative and univalent universe?
- Applications.
- Weak ∞-groupoidal realizability. Want a weaker notion of interval; $[0, 1] \in \mathbf{Top}$ should be an example.

Groupoidal realizability over cubical λ-calculus

[Scott '80] shows how to organise the untyped λ-calculus into a CCC with universal object: take the Karoubi envelope of the monoid of λ-terms t satisfying $t = \lambda x.tx$.

Can one build a realizer category (or similar) out of the untyped cubical λ-calculus?

Instead, one can attempt to do groupoidal realizability directly over the untyped cubical λ-calculus. Replace ΠU with the "fundamental groupoid of the untyped cubical λ-calculus":

- Objects: terms $\Gamma \vdash a\ [\cdot]$ in empty dimension context,
- Morphisms: homotopy classes of terms $\Gamma \vdash \alpha\ [i]$ in singleton context.