Towards coherence theorems for equational extensions of type theories

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Introduction

1. Introduction

2. Generalized algebraic theories with homotopy relations
(General statement of conservativity)

3. Partial saturation
(Proof strategy for conservativity)

4. Strict Rezk completions
(Work in progress)

Weak and Strict type theories

Trade-offs between type theories with more or less definitional equalities.

Weaker theories:

- More models.
- Type theories without any definitional equalities are cofibrant in categories of theories.

Stronger theories:

- Shorter internal proofs and constructions.
- Definitional equalities are automatically coherent.
  $\rightsquigarrow$ Avoids “higher transport hell”.

Conservativity/Coherence/Strictification theorems should provide interpretations of stronger type theories in weaker models.

Weakenings/Strengthenings of HoTT

Weakenings of HoTT:

- Weakly computational identity types;
- Weak Tarski Universes;
- Weak/Propositional/Objective Type Theory.

Strengthenings of HoTT:

- Definitional semiring laws for $\mathbb{N}$;
- Universe SProp of strict propositions (definitionally proof-irrelevant);
- Strict 1-groupoid laws for identity types;
- Universes of definitional rings, definitional categories, etc.;
- ...

Extensions

These extensions factor in two steps:

1. Add new constants.

$$\_ + \_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N},$$
$$\mathsf{plus}_0 : \forall n, n + 0 \simeq n,$$
$$\mathsf{plus}_1 : \forall n\ m, n + S(m) \simeq S(n + m),$$
$$\mathsf{plus}_2 : \forall n, n + m \simeq m + n,$$
$$\ldots$$

   The total type of these constants should be **contractible**.

2. **Equational extension:** Add new definitional equalities.

$$n + 0 = n, \qquad n + S(m) = S(n + m), \qquad n + m = m + n,$$
$$\mathsf{plus}_1 = \mathsf{refl}, \qquad \mathsf{plus}_0 = \mathsf{refl}, \qquad \mathsf{plus}_2 = \mathsf{refl}, \qquad \ldots$$

Hofmann's conservativity theorem

Uniqueness of Identity Proofs

$$\frac{p : x \simeq x}{\mathsf{uip}(p) : p \simeq \mathsf{refl}}$$

Equality reflection

$$\frac{p : x \simeq y}{x = y \qquad p = \mathsf{refl}}$$

Intensional Type Theory has UIP (and function extensionality).

Extensional Type Theory has equality reflection.

Theorem (Hofmann)

ETT is conservative over ITT.

Why is UIP needed ?

Assume we have in the source theory:

$$f : A \to B,$$
$$a : A'.$$

such that $|A| = |A'|$ in the target theory.

The application $f(a)$ is well-typed in the target, but not in the source.

$\rightsquigarrow$ Translate $f(a)$ to $f(\text{transport}(p, a))$ where $p : A \simeq_{\mathcal{U}} A'$ is a path in the universe.

With UIP, the choice of p does not matter.

Without UIP, all choices need to be **coherent**.

(Even with UIP, choices matter if we choose equivalences $A \cong A'$ instead of paths)

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Informal proof strategy

Given equational extension $\mathcal{T} \to \mathcal{T}_E$, define a factorization

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\qquad} & \mathcal{T}_E \\ & \searrow \quad \nearrow & \\ & \mathcal{T}_{\widehat{E}} & \end{array}$$

$\mathcal{T}_{\widehat{E}}$ should have:

- A notion of **coherent** equivalence/identifications;
- **Formal transports** over these coherent equivalences/identifications.

Conservativity (property of $\mathcal{T} \to \mathcal{T}_E$) should follow from coherence (property of $\mathcal{T}_{\widehat{E}}$).

Coherence: any two parallel coherent equivalences/identifications are coherently identified.
$\rightsquigarrow$ Choices of coherent equivalences don't matter.

Generalized algebraic theories with homotopy relations

Categories with Families

A category with family (CwF) $\mathcal{C}$ has:

- **Contexts/Objects:** $\Gamma \in \mathcal{C}$
- **Substitutions/Morphisms:** $\gamma \in \mathcal{C}(\Delta, \Gamma)$
- **Types:** $(\Gamma \vdash A \text{ type}) \in \mathcal{C}$, or $A : \mathcal{C}.\mathsf{Ty}(\Gamma)$.
- **Terms:** $(\Gamma \vdash a : A) \in \mathcal{C}$, or $a : \mathcal{C}.\mathsf{Tm}(\Gamma, A)$.

Structured CwFs (should) correspond to classes of 1- or ∞- categories and algebraic theories.

- Σ**-CwFs** $\rightsquigarrow$ **clans**; (GATs)
- (Σ, Eq)**-CwFs** $\rightsquigarrow$ **finitely complete** 1**-categories**; (EATs)
- (Σ, Id)**-CwFs** $\rightsquigarrow$ **finitely complete** ∞**-categories**; (∞-EATs)
- $(\Sigma, \Pi_{\mathsf{rep}})$**-CwFs** $\rightsquigarrow$ **representable map clans**; (SOGATs)
- ...

Generalized Algebraic Theories

A GAT is like an algebraic theory, except that sorts can be dependent.

Example: the GAT of preorders has:

- Two sorts (the underlying set and the relation).
- Two operations (reflexivity and transitivity).
- One equation ($\_ < \_$ is a family of propositions).

$$\begin{aligned}
&\mathsf{Ob} : \mathrm{Set}, \\
&_ < _ : \mathsf{Ob} \to \mathsf{Ob} \to \mathrm{Set}, \\
&\mathsf{refl} : x < x, \\
&\mathsf{trans} : x < y \to y < z \to x < z, \\
&\forall (f, g : x < y), f = g.
\end{aligned}$$

Functorial semantics of GATs

A GAT admits a classifying Σ-CwF presented by:

- Generating types (sorts);
- Generating terms (operations);
- Equations between terms.

The GAT $\mathcal{T}_{\mathbf{Preord}}$ is the Σ-CwF generated by:

$$
\begin{aligned}
&(1 \vdash \mathrm{Ob}\ \mathrm{type}),\\
&(x, y : \mathrm{Ob} \vdash x < y\ \mathrm{type}),\\
&(x : \mathrm{Ob} \vdash \mathrm{refl} : x < x),\\
&(x, y, z : \mathrm{Ob}, p : x < y, q : y < z \vdash \mathrm{trans} : x < z),\\
&(x, y : \mathrm{Ob}, p : x < y, q : x < y \vdash p = q).
\end{aligned}
$$

Functorial semantics of GATs

A contextual model of $\mathcal{T}$ is a Σ-CwF morphism $\mathcal{T} \to \mathbf{Set}$.
(Interpretation of the sorts and operations as families and functions)

Given $\mathcal{M} : \mathcal{T}_{\mathbf{Preord}} \to \mathbf{Set}$,

$$\begin{aligned}
&\mathcal{M}(\mathsf{Ob}) : \mathrm{Set}, \\
&\mathcal{M}(\mathsf{Hom}) : \mathcal{M}(\mathsf{Ob}) \times \mathcal{M}(\mathsf{Ob}) \to \mathrm{Set}, \\
&\mathcal{M}(\mathsf{refl}) : (x : \mathcal{M}(\mathsf{Ob})) \to \mathcal{M}(\mathsf{Hom})(x, x), \\
&\ldots
\end{aligned}$$

A morphism between $\mathcal{M}, \mathcal{N} : \mathcal{T} \to \mathbf{Set}$ is a natural transformation $\mathcal{M} \Rightarrow \mathcal{N}$.

A (generalized) model of $\mathcal{T}$ is a category $\mathcal{C}$, along with a Σ-CwF morphism $\mathcal{T} \to \mathbf{Psh}(\mathcal{C})$.

There is a (generalized) model $よ : \mathcal{T} \to \mathbf{Psh}(\mathcal{T})$.

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Morphisms of GATs

Other GATs: $\mathcal{T}_{\mathbf{Poset}}$, $\mathcal{T}_{\mathbf{Cat}}$, $\mathcal{T}_{\mathbf{MonCat}}$, $\mathcal{T}_{\mathbf{StrMonCat}}$, etc.

Remark: $\mathcal{T}_{\mathbf{Cat}}$ has three generating sorts: Ob, Hom and EqHom!

Equality between morphisms is part of the "language of categories", but equality between objects is not.

GAT morphisms are morphisms between their classifying Σ-CwF.

$$\mathcal{T}_{\mathbf{Preord}} \longrightarrow \mathcal{T}_{\mathbf{Poset}}$$

$$\mathcal{T}_{\mathbf{Cat}} \longrightarrow \mathcal{T}_{\mathbf{MonCat}} \longrightarrow \mathcal{T}_{\mathbf{StrMonCat}}$$

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Why look at the classifying Σ-CwF ?

There is a fully faithful functor

$$\mathbf{0}_{\mathcal{T}}[-] : \mathcal{T} \to \mathbf{Mod}_{\mathcal{T}}^{\mathsf{op}}.$$

Its essential image consists of the **finitely generated models of $\mathcal{T}$**.

$$\mathbf{0}_{\mathcal{T}_{\mathsf{Cat}}}[x : \mathrm{Ob}, y : \mathrm{Ob}, f : \mathrm{Hom}(x, y)] = \{x \xrightarrow{f} y\}$$

Every model is equivalent to a freely generated model. (Small object argument)

Every freely generated model is a filtered colimit of finitely generated models. ($\mathcal{T}$ is finitary)

$\rightsquigarrow$ Looking at $\mathcal{T}$ gives information about all models.

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Second-order generalized algebraic theories

The difference between first-order and second-order is not important in this talk.

Second-order generalized algebraic theories also have **representable** sorts.

Example: most type theories are SOGATs with two sorts.

$$(1 \vdash \mathsf{Ty}\ \mathsf{type}),$$
$$(A : \mathsf{Ty} \vdash \mathsf{Tm}(A)\ \mathsf{type}_{\mathsf{rep}}).$$

$\mathsf{Tm}(A)$ being representable means that we have context extensions and term variables.

Example of SOGATs: $\mathcal{T}_{\mathsf{Id}}$, $\mathcal{T}_{\mathsf{Id}_s}$, $\mathcal{T}_{\Sigma}$, $\mathcal{T}_{\Sigma,\Pi_{\mathsf{rep}}}$, $\mathcal{T}_{\mathsf{ITT}}$, $\mathcal{T}_{\mathsf{ETT}}$, $\mathcal{T}_{\mathsf{HoTT}}$, etc.

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Trivial fibrations

Let $F : \mathcal{N} \to \mathcal{M}$ be a morphism of models of a GAT $\mathcal{T}$.

Definition

The map F is a trivial fibration if for every generating sort $(\partial S \vdash S \text{ type}) \in \mathcal{T}$, we have:

Strict lifting For every $(\sigma : \partial S) \in \mathcal{N}$ and $(x : S(F(\sigma))) \in \mathcal{M}$, there is $(x_0 : S(\sigma)) \in \mathcal{N}$ such that $F(x_0) = x$.

In other words, the action of F on every sort is surjective.

Trivial fibrations in **Cat** are functor that are surjective objects and fully faithful.

Theorem (Hofmann's conservativity theorem)

The morphism $\mathbf{0}_{\text{ITT}} \to \mathbf{0}_{\text{ETT}}$ is a trivial fibration in $\mathbf{Mod}_{\mathcal{T}_{\text{ITT}}}$.

Homotopy relations on a GAT

For every generating sort $(\partial S \vdash S \text{ type}) \in \mathcal{T}$,

$$(\sigma : \partial S, x : S(\sigma), y : S(\sigma) \vdash x \sim_{S(\sigma)} y \text{ type}) \in \mathcal{T},$$
$$(\sigma : \partial S, x : S(\sigma) \vdash \mathsf{hrefl} : x \sim_{S(\sigma)} x) \in \mathcal{T}.$$

Example for $\mathcal{T}_{\mathbf{Cat}}$:

$$\begin{aligned}
(x \sim_{\mathsf{Ob}} y) &\triangleq \mathsf{Iso}(x, y),\\
(f \sim_{\mathsf{Hom}(x,y)} g) &\triangleq \mathsf{EqHom}(f, g),\\
(p \sim_{\mathsf{EqHom}(f,g)} q) &\triangleq \mathbf{1}.
\end{aligned}$$

Weak equivalences

Let $F : \mathcal{N} \to \mathcal{M}$ be a morphism of models of $\mathcal{T}$.

Definition

The map F is a weak equivalence if for every generating sort $(\partial S \vdash S \text{ type}) \in \mathcal{T}$, we have:

Weak lifting For every $(\sigma : \partial S) \in \mathcal{N}$ and $(x : S(F(\sigma))) \in \mathcal{M}$, there is $(x_0 : S(\sigma)) \in \mathcal{N}$ and $(p : F(x_0) \sim x) \in \mathcal{M}$.

In other words, the action of F on every sort is surjective up to homotopy.

Example for $\mathcal{T}_{\mathbf{Cat}}$:

- Weak lifting for Ob: the functor F is essentially surjective;
- Weak lifting for Hom: the functor F is full;
- Weak lifting for EqHom: the functor F is faithful.

Conservativity

Assume that $\mathcal{T}_1$ is equipped with homotopy relations.

Definition

A morphism $F : \mathcal{T}_1 \to \mathcal{T}_2$ of (SO)GATs is a Morita equivalence if it is a weak equivalence in $\mathbf{Mod}_{\mathcal{T}_1}$.

Equivalently, $\mathbf{0}_{\mathcal{T}_1}[\Gamma] \to \mathbf{0}_{\mathcal{T}_2}[F(\Gamma)]$ is a weak equivalence in $\mathbf{Mod}_{\mathcal{T}_1}$ for every $\Gamma \in \mathcal{T}_1$.

Equivalently, $\eta_{\mathcal{C}} : \mathcal{C} \to F^*(F_!(\mathcal{C}))$ for every cofibrant $\mathcal{C} \in \mathbf{Mod}_{\mathcal{T}_1}$.

Summary so far

- Focus on the classifying Σ-CwF (or $(\Sigma, \Pi_{\mathsf{rep}})$-CwF) of (SO)GATs.
- GAT $\rightsquigarrow$ Notion of trivial fibration.
- GAT with homotopy relations $\rightsquigarrow$ Notion of weak equivalence (also fibrations).
  For $\mathcal{T}_{\mathbf{Cat}}$: classes of maps of the canonical model structure on **Cat**.
  For $\mathcal{T}_{\mathsf{Id}}$: classes of maps of the left semi-model structure on $\mathbf{CwF}_{\mathsf{Id}}$.
- A morphism $\mathcal{T}_1 \to \mathcal{T}_2$ of (SO)GATs is an equivalence if it is a weak equivalence in $\mathbf{Mod}_{\mathcal{T}_1}$.

Partial saturation

1. Introduction

2. Generalized algebraic theories with homotopy relations
(General statement of conservativity)

3. Partial saturation
(Proof strategy for conservativity)

4. Strict Rezk completions
(Work in progress)

Equational extensions

Let $\mathcal{T}$ be a GAT equipped with homotopy relations.

Let E be a collection of homotopies in $\mathcal{T}$.

$$E \subseteq \{(\Gamma, \sigma, x, y, p) \mid (\Gamma \vdash p : x \sim_{S(\sigma)} y) \in \mathcal{T}\}.$$

The equational extension $\mathcal{T} \to \mathcal{T}_E$ is the extension of $\mathcal{T}$ by equations

$$x = y,$$

$$p = \mathsf{hrefl} \qquad \text{(constant homotopy)}$$

for all $(\Gamma \vdash p : x \sim_{S(\sigma)} y) \in E$.

Example (for $\mathcal{T}_{\mathbf{MonCat}} \to \mathcal{T}_{\mathbf{StrMonCat}}$):

$$E = \{(x, y, z : \mathsf{Ob} \vdash \alpha_{x,y,z} : (x \otimes (y \otimes z)) \cong ((x \otimes y) \otimes z)), \lambda, \rho, (\text{pentagon}), (\text{triangle})\}.$$

Partial saturation

Let $\mathcal{C}$ be a (Σ, Id)-CwF equipped with an internal model $\mathcal{M} : \mathcal{T} \to \mathcal{C}$.

We have maps

$$\text{id-to-hpty}_S : (x \simeq_{S(\sigma)} y) \to (x \sim_{S(\sigma)} y).$$

Definition

We say that $\mathcal{C}$ is saturated (or that $\mathcal{M}$ is univalent) if the maps id-to-hpty_S are equivalences.

A Σ-CwF morphism $\mathcal{T}_{\mathbf{Cat}} \to \mathcal{C}$ is an internal category in $\mathcal{C}$;
It is univalent if it is an internal univalent category in $\mathcal{C}$.

Partial saturation

Let $\mathcal{C}$ be a (Σ, Id)-CwF equipped with an internal model $\mathcal{M} : \mathcal{T} \to \mathcal{C}$.

We have maps

$$\text{id-to-hpty}_S : (x \simeq_{S(\sigma)} y) \to (x \sim_{S(\sigma)} y).$$

Definition

We say that $\mathcal{C}$ is partially saturated with respect to E if we have

$$(\Gamma \vdash \widehat{p} : x \simeq_{S(\sigma)} y) \in \mathcal{C},$$
$$(\Gamma \vdash \widetilde{p} : \text{id-to-hpty}_S(\widehat{p}) \simeq p) \in \mathcal{C},$$

for every $(\Gamma \vdash p : x \sim_{S(\sigma)} y) \in E$.

Write $\mathcal{T}^{\infty}_{\widehat{E}}$ for the initial (Σ, Id)-CwF equipped with a partially saturated internal model.

Partial saturation

Example for $\mathcal{T} = \mathcal{T}_{\textbf{MonCat}}$ and

$$E = \{(x, y, z : \mathsf{Ob} \vdash \alpha_{x,y,z} : (x \otimes (y \otimes z)) \cong ((x \otimes y) \otimes z)), \lambda, \rho, (\text{pentagon}), (\text{triangle})\}.$$

$\mathcal{T}^{\infty}_{\widehat{E}}$ has (weak) identity types and

$$\begin{aligned} &\widehat{\alpha}_{x,y,z} : (x \otimes (y \otimes z)) \simeq_{\mathsf{Ob}} ((x \otimes y) \otimes z)), \\ &\widetilde{\alpha}_{x,y,z} : \text{id-to-hpty}_{\mathsf{Ob}}(\widehat{\alpha}_{x,y,z}) \simeq \alpha_{x,y,z}, \\ &\dots \end{aligned}$$

$\rightsquigarrow$ Identifications $x \simeq_{\mathsf{Ob}} y$ are approximately compositions of associators and unitors.

Main diagram

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_E \\ \downarrow & & \downarrow \\ \mathcal{T}^{\infty}_{\widehat{E}} & \longrightarrow & \mathcal{T}^{1}_{E} \end{array}$$

- $\mathcal{T}^{\infty}_{\widehat{E}}$: $\mathbf{CwF}_{\Sigma,\mathsf{Id}}$ is obtained by adding identity types + partial saturation to $\mathcal{T}$.
- $\mathcal{T}^{1}_{E}$: $\mathbf{CwF}_{\Sigma,\mathsf{Eq}}$ is obtained by adding equality reflection to $\mathcal{T}^{\infty}_{E}$.
  Or equivalently by adding equality types to $\mathcal{T}_E$.

Main diagram: right map

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_E \\ \downarrow & & \downarrow \\ \mathcal{T}^{\infty}_{\widehat{E}} & \longrightarrow & \mathcal{T}^1_E \end{array}$$

The CwF morphism $L : \mathcal{T}_E \to \mathcal{T}^1_E$ is always bijective on terms.

This is a canonicity result for $\mathcal{T}^1_E$:
Terms of $\mathcal{T}^1_E$ over contexts of the form $L(-)$ compute to terms of the form $L(-)$.

Proof can be given in the internal language of $\mathbf{Psh}(\mathbf{Ren}(\mathcal{T}_E))$.

Main diagram: left map

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_E \\ \downarrow & & \downarrow \\ \mathcal{T}^{\infty}_{\widehat{E}} & \longrightarrow & \mathcal{T}^1_E \end{array}$$

The CwF morphism $K : \mathcal{T} \to \mathcal{T}^{\infty}_{\widehat{E}}$ should be a weak equivalence in $\mathbf{Mod}_{\mathcal{T}}$, when $\mathcal{T}$ is well-behaved (but independently of E).

This is a homotopy canonicity property for $\mathcal{T}^{\infty}_{\widehat{E}}$:
Terms of $\mathcal{T}^{\infty}_{\widehat{E}}$ over contexts $K(-)$ compute, up to homotopy, to terms of the form $K(-)$.

The proof should be given in the internal language of $\mathbf{Psh}_{\infty}(\mathbf{Ren}(\mathcal{T}))$?

Needs ∞-groupoid structure of the components of $\mathcal{T}$ and $\mathcal{T}^{\infty}_{\widehat{E}}$.

Main diagram: bottom map

$$
\begin{array}{ccc}
\mathcal{T} & \longrightarrow & \mathcal{T}_E \\
\downarrow & & \downarrow \\
\mathcal{T}^{\infty}_{\widehat{E}} & \color{red}{\longrightarrow} & \mathcal{T}^{1}_{E}
\end{array}
$$

The CwF morphism $\mathcal{T}^{\infty}_{\widehat{E}} \to \mathcal{T}^{1}_{E}$ freely adds equality reflection.

This is similar to the extension from ITT to ETT.

Theorem

If $\mathcal{T}^{\infty}_{\widehat{E}}$ is merely 0-truncated, then $\mathcal{T}^{\infty}_{\widehat{E}} \to \mathcal{T}^{1}_{E}$ is a trivial fibration.

Merely 0-truncated: for every $(\Gamma \vdash p : x \simeq x) \in \mathcal{T}^{\infty}_{\widehat{E}}$, there merely exists $(\Gamma \vdash \mathrm{uip}(p) : p \simeq \mathrm{refl}) \in \mathcal{T}^{\infty}_{\widehat{E}}$.

Main diagram: bottom map

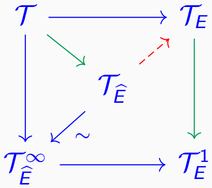
$$\begin{array}{ccc}
\mathcal{T} & \longrightarrow & \mathcal{T}_E \\
\downarrow & & \downarrow \\
\mathcal{T}^{\infty}_{\widehat{E}} & \longrightarrow & \mathcal{T}^{1}_{E}
\end{array}$$

Problem: $\mathcal{T}^{\infty}_{\widehat{E}}$ is almost never 0-truncated.
(Consider $(x : A, p : x \simeq x \vdash p : x \simeq x) \in \mathcal{T}^{\infty}_{\widehat{E}}$.)

Solution: 0-truncation over the image of K is enough.

Merely 0-truncated relatively to K: for every $(K(\Gamma) \vdash p : x \simeq x) \in \mathcal{T}^{\infty}_{\widehat{E}}$, there merely exists $(K(\Gamma) \vdash \mathsf{uip}(p) : p \simeq \mathsf{refl}) \in \mathcal{T}^{\infty}_{\widehat{E}}$.

Main diagram: factorization



Consider factorization $\mathcal{T} \to \mathcal{T}_{\widehat{E}} \to \mathcal{T}_{\widehat{E}}^{\infty}$. (in some orthogonal FS)
The CwF $\mathcal{T}_{\widehat{E}}$ is the restriction of $\mathcal{T}_{\widehat{E}}^{\infty}$ to contexts/types that do not contain identity types.
The map $\mathcal{T}_{\widehat{E}} \to \mathcal{T}_{\widehat{E}}^{\infty}$ is bijective on terms.

Solution to lifting problem gives $\mathcal{T}_{\widehat{E}} \to \mathcal{T}_E$.
Replaces bottom and right maps.

Theorem

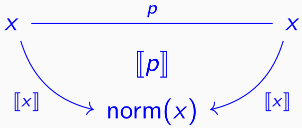
If $\mathcal{T}_{\widehat{E}}^{\infty}$ is merely 0-truncated relatively to K, then $\mathcal{T}_{\widehat{E}} \to \mathcal{T}_E$ is a trivial fibration.

0-truncatedness from normalization

0-truncatedness follows from homotopy normalization results:

Interpret element $x : A$ as a path $[\![x]\!] : x \simeq \mathsf{norm}(x)$.
Interpret path $p : x \simeq y$ as a dependent path $[\![p]\!] : [\![x]\!] \simeq [\![y]\!]$ over p.

When $p : x \simeq x$, $[\![p]\!]$ implies that $p \simeq \mathsf{refl}$.

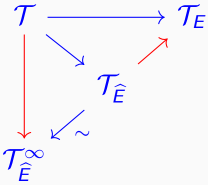


Strict normalization proof for $\mathcal{T}_E$ takes place in $\mathbf{Psh}(\mathbf{Ren}(\mathcal{T}_E))$.

Homotopy normalization proof for $\mathcal{T}^{\infty}_{\widehat{E}}$ should take place in $\mathbf{Psh}_{\infty}(\mathbf{Ren}(\mathcal{T}_{\widehat{E}}))$!

Needs the ∞-groupoid structure of the components of $\mathcal{T}^{\infty}_{\widehat{E}}$!

Factorization:



- $\mathcal{T} \to \mathcal{T}^{\infty}_{\widehat{E}}$ being a weak equivalence is homotopy canonicity for $\mathcal{T}^{\infty}_{\widehat{E}}$.
- $\mathcal{T}_{\widehat{E}} \to \mathcal{T}_E$ being a trivial fibration follows from homotopy normalization for $\mathcal{T}^{\infty}_{\widehat{E}}$.

How do we prove homotopy canonicity/normalization for $\mathcal{T}^{\infty}_{\widehat{E}}$?

Strict Rezk completions

Strict canonicity

Strict canonicity for e.g. MLTT can be proven using logical predicates:

Interpret $A : \mathcal{S}.\mathsf{Ty}(1)$ as a family $[\![A]\!] : \mathcal{S}.\mathsf{Tm}(1, A) \to \mathrm{Set}$.
Interpret $a : \mathcal{S}.\mathsf{Tm}(1, A)$ as an element $[\![a]\!] : [\![A]\!](a)$.

For specific types, an element of $[\![A]\!](a)$ proves that a is canonical.
e.g. $[\![\mathsf{Bool}]\!](b) := (b = \mathsf{true}) + (b = \mathsf{false})$.

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Homotopy canonicity

Now assume that $\mathcal{S}$ is a CwF with identity types + univalence/saturation/partial saturation.

$[\![A]\!] : \mathcal{S}.\mathsf{Tm}(1, A) \to \mathrm{Set}$ should be replaced by $[\![A]\!] : \mathcal{S}.\mathsf{Tm}(1, A) \to \infty\mathsf{Grp}$.

Problem: $\mathcal{S}.\mathsf{Tm}(1, A)$ is a set, not an ∞-groupoid.

$\rightsquigarrow$ We need to replace $\mathcal{S}.\mathsf{Tm}(1, A)$ by an ∞-groupoid $\overline{\mathcal{S}}.\mathsf{Tm}(1, A)$.

$\rightsquigarrow$ We need to replace all components of $\mathcal{S}$ by ∞-groupoids/∞-functors. This should be compatible with the strict substitution of $\mathcal{S}$.

Solution ? $\overline{\mathcal{S}}$ should be the "strict Rezk completion" of $\mathcal{S}$ in cartesian cubical sets.

If $\mathcal{C}$ is a category in HoTT, its Rezk completion $\overline{\mathcal{C}}$ has the correct ∞-groupoid of objects.

A strict Rezk completion $\overline{\mathcal{S}}$ (if it exists) would have the correct ∞-groupoids of types/terms.

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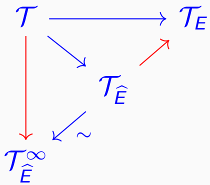
Conclusion

- Conservativity problems can be stated for morphisms of generalized algebraic theories.
- Instead of looking at all models, we can look at the classifying CwF of a theory.
- Conservativity should follow from homotopy canonicity and homotopy normalization.



https://arxiv.org/abs/2304.10343

Strict Rezk completion of preorders

Let X be a strict preorder. Define a strict preorder $\overline{X}$, freely generated by:

$$i : X \to \overline{X},$$
$$\mathsf{Glue}_{\mathsf{Ob}} : (x : \overline{X})(\alpha : \mathsf{Cof})((y, e) : [\alpha] \to \Sigma_y(x \cong y)) \to \{\overline{X} \mid \alpha \hookrightarrow y\},$$
$$\mathsf{glue}_{\mathsf{Ob}} : (x : \overline{X})(\alpha : \mathsf{Cof})((y, e) : [\alpha] \to \Sigma_y(x \cong y)) \to \{x \cong \mathsf{Glue}_{\mathsf{Ob}}(x, (y, e)) \mid \alpha \hookrightarrow e\},$$
$$\mathsf{Glue}_{\mathsf{Hom}} : (f : x <_{\overline{X}} y)(\alpha : \mathsf{Cof})(g : [\alpha] \to x <_{\overline{X}} y) \to \{x <_{\overline{X}} y \mid \alpha \hookrightarrow g\}.$$

Theorem

If X is cofibrant in **Preord** *and its components are fibrant (Kan), then the preorder $\overline{X}$ is a strict Rezk completion of X.*

(The hard part is the fibrancy of $\overline{X}$)