Causal Coverage in Ordered Locales

PSSL 109

Nesta vd Schaaf
with Chris Heunen

Inria

THE UNIVERSITY OF EDINBURGH

{arXiv: 2406.15406}

17-11-24

Idea

$$(U_i) \in J(U) \qquad\qquad (A_i) \in \bar{J}(U)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\bigvee U_i = U \qquad\qquad ???$$

{arXiv: 2406.15406}

Ordered Locales

"Ordered Locales"
JPAA 228(7), 2024

$(X, \trianglelefteq)$

Ordered locales

"Ordered Locales"
JPAA 228(7), 2024

$(X, \trianglelefteq)$ + axiom

locale

preorder on $\mathcal{O}X$

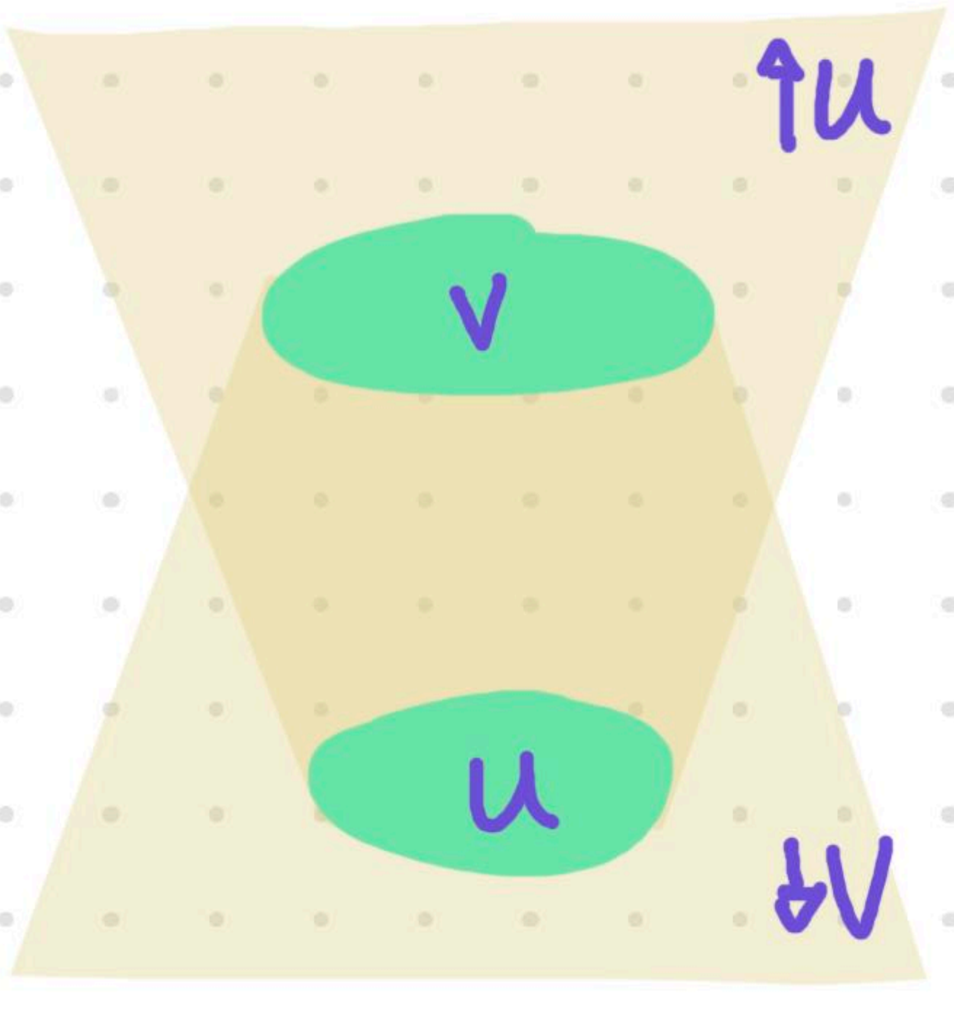
Ordered locales

"Ordered Locales"
JPAA 228(7), 2024

$(X, \trianglelefteq)$ + axiom

X: locale

$\trianglelefteq$: preorder on $\mathcal{O}X$



Ordered Locales

"Ordered Locales"
JPAA 228(7), 2024

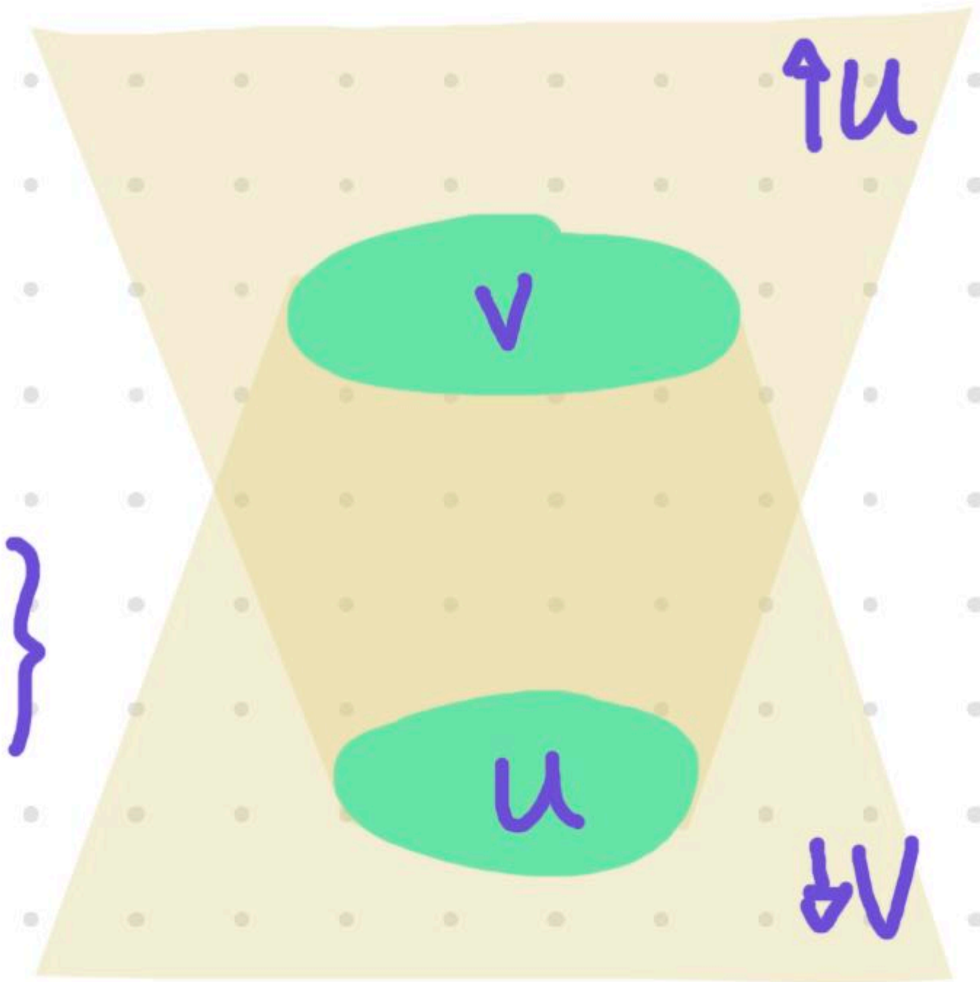
$(X, \trianglelefteq)$ + axiom

locale

preorder on $\mathcal{O}X$

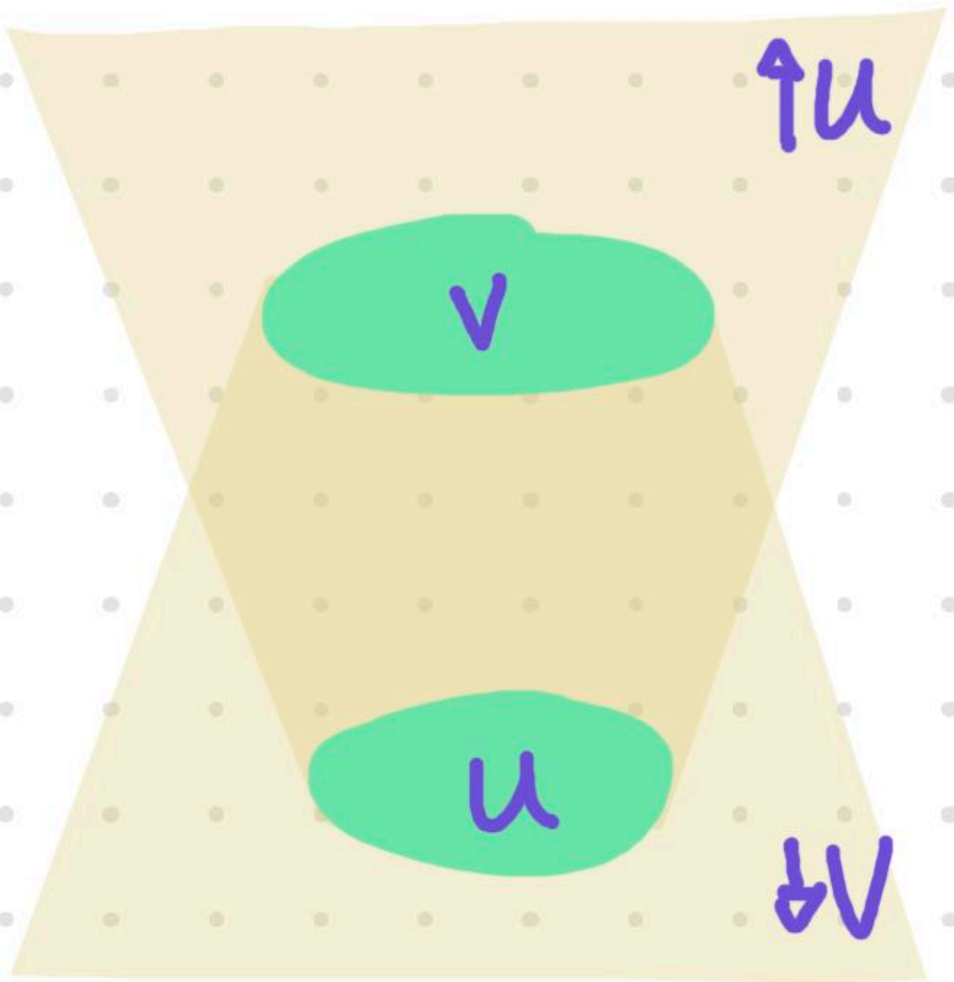
$$\uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}$$

monad



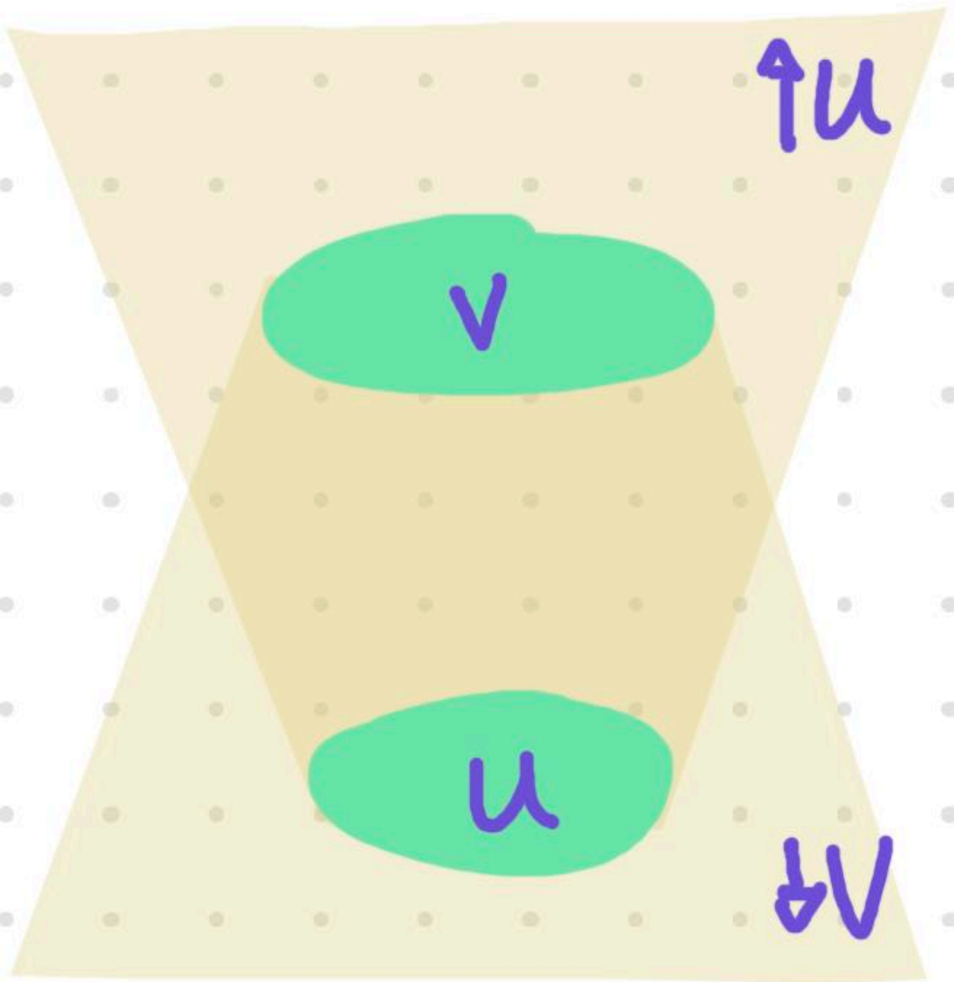
Ordered locales

$\trianglelefteq \quad \leftrightsquigarrow \quad \Uparrow, \Downarrow$



Ordered locales

$$\trianglelefteq \quad \leftrightsquigarrow \quad \underbrace{\uparrow \,,\ \downarrow}_{\text{no relation}}$$



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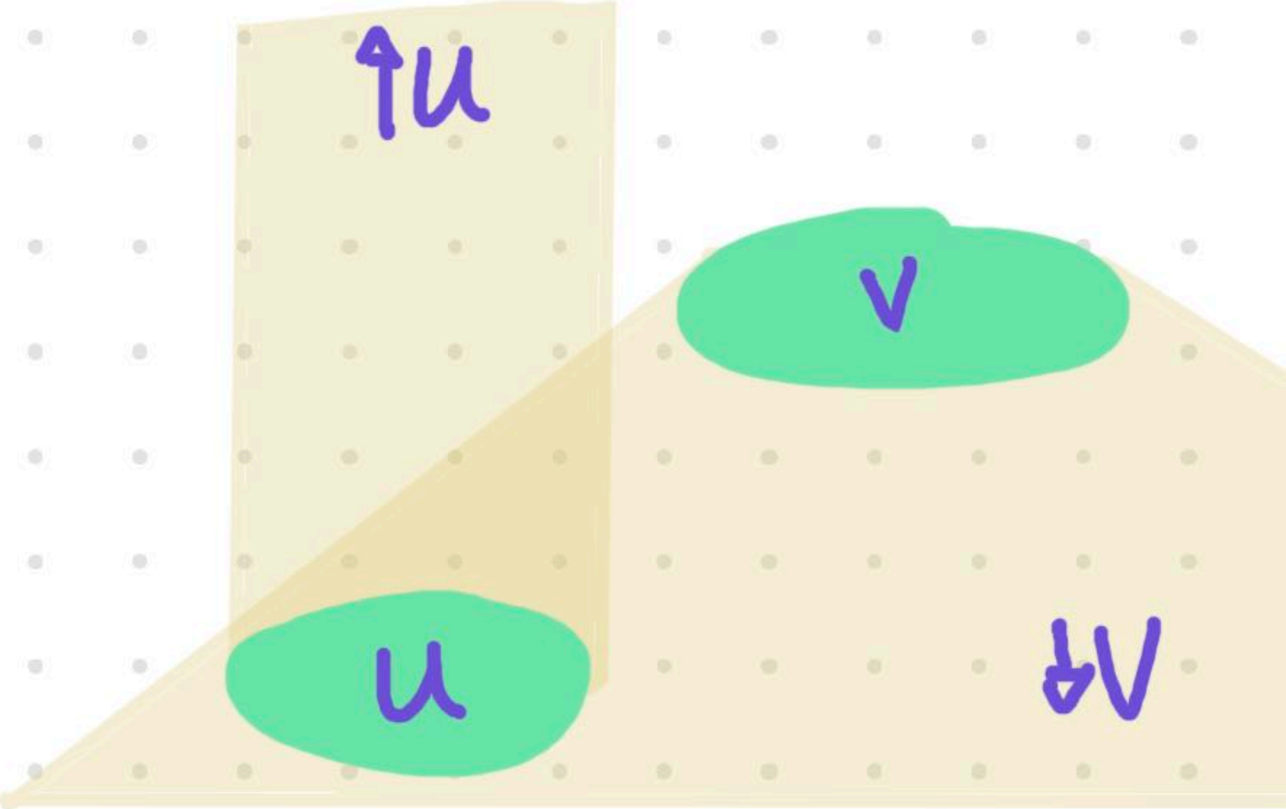
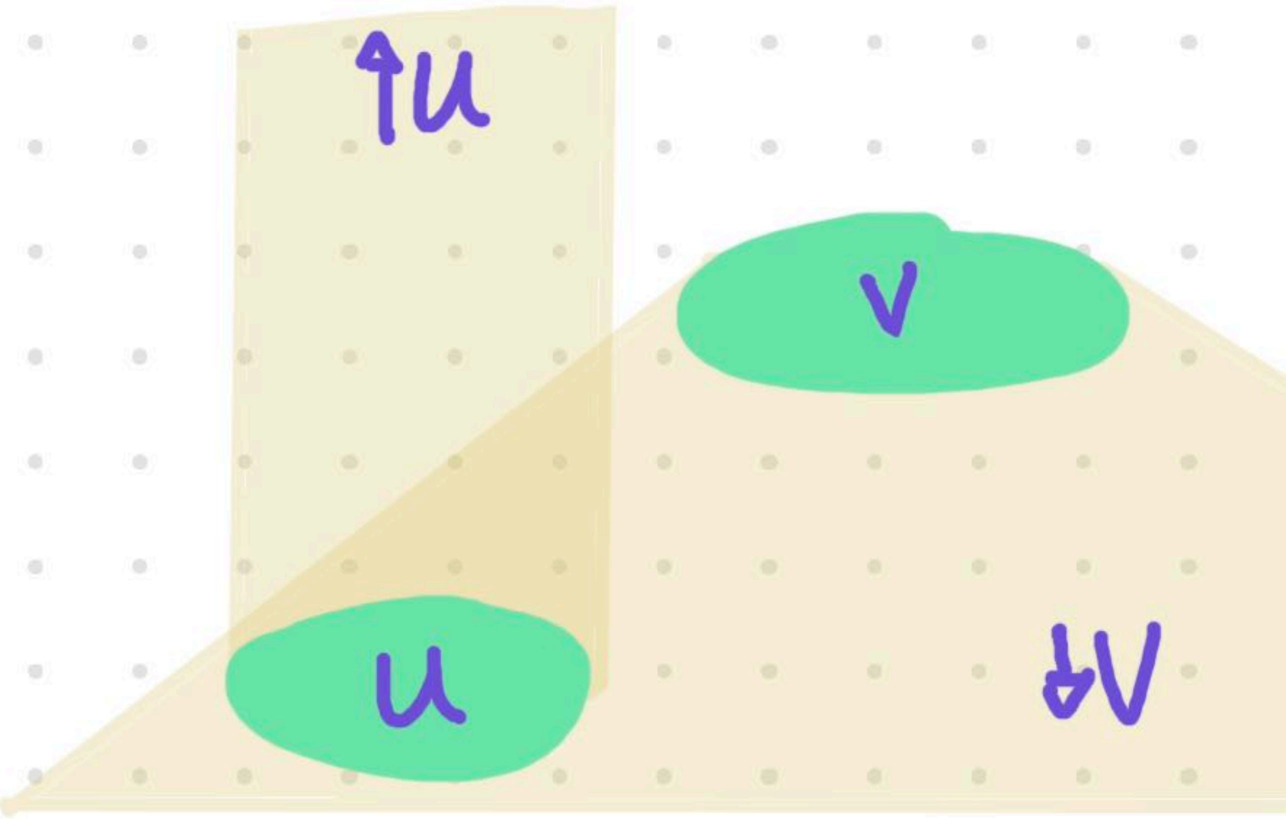


Ordered locales

$$\trianglelefteq \;\leftrightsquigarrow\; \underbrace{\uparrow\ ,\ \downarrow}_{\text{no relation}}$$

parallel ordered:

$$\uparrow U \wedge V = \emptyset$$
$$\Updownarrow$$
$$U \wedge \downarrow V = \emptyset$$



Idea

$$(U_i) \in J(U)$$

$$\Updownarrow$$

$$\bigvee U_i = U$$

Idea

$$(A_i) \in \bar{J}(U)$$

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???

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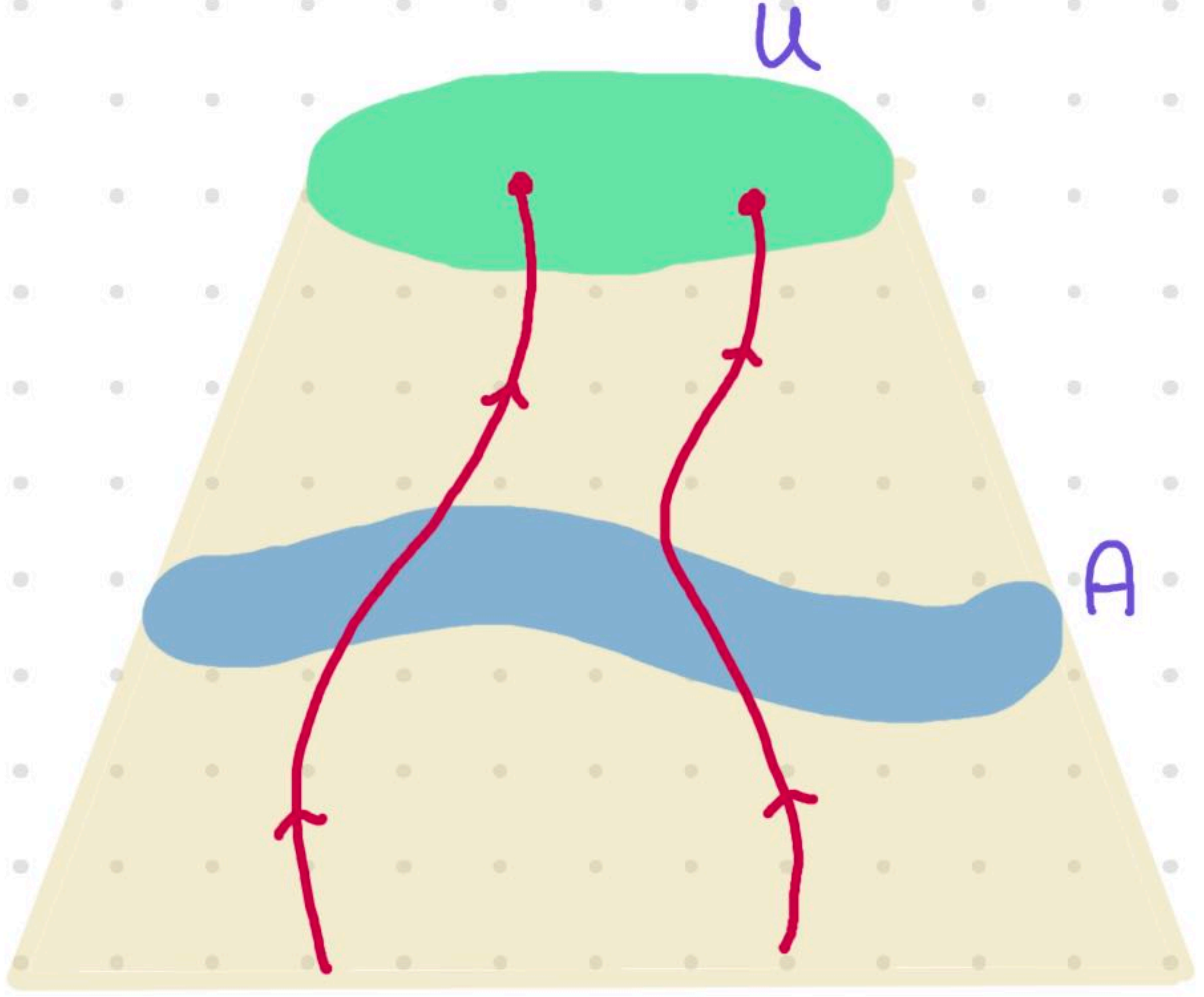
???

[Christensen-Crane '05]



Paths

$$P_1 \trianglelefteq \cdots \trianglelefteq P_T$$

non-empty

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P_1

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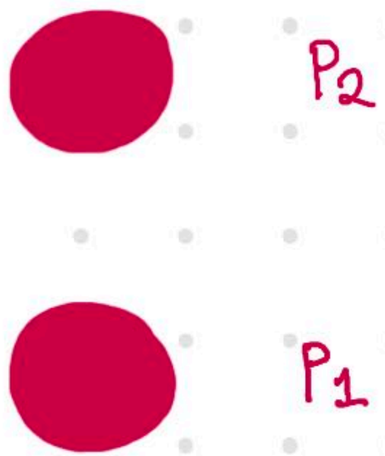

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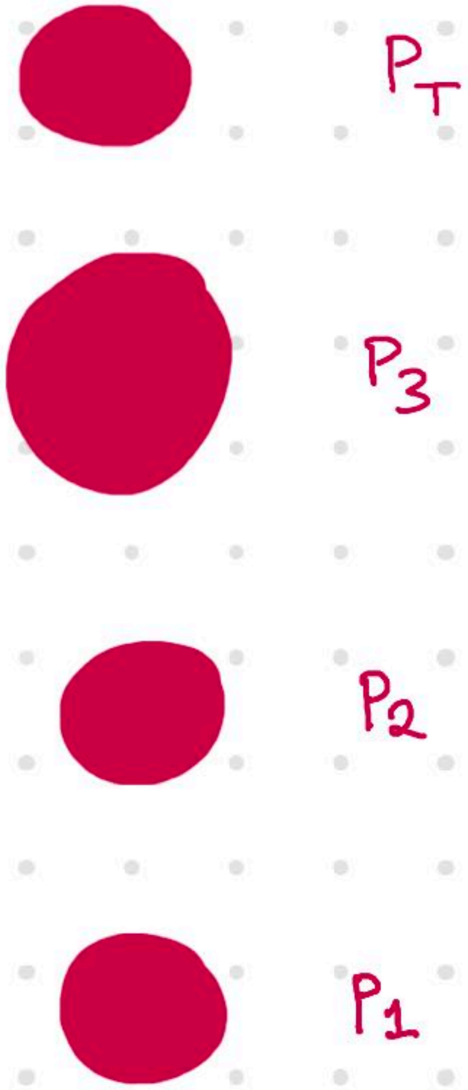
non-empty

Paths

$$\underbrace{P_1}_{\text{non-empty}} \trianglelefteq \cdots \trianglelefteq P_T$$



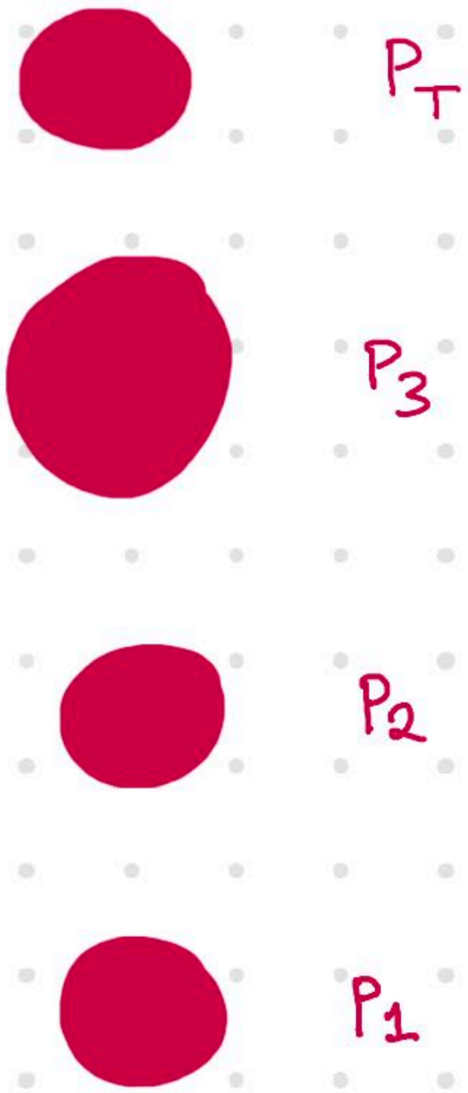
Paths

$$p_1 \trianglelefteq \cdots \trianglelefteq p_T$$

refinement

$$q \Subset p$$

$$\forall n \exists m : q_m \subseteq p_n$$



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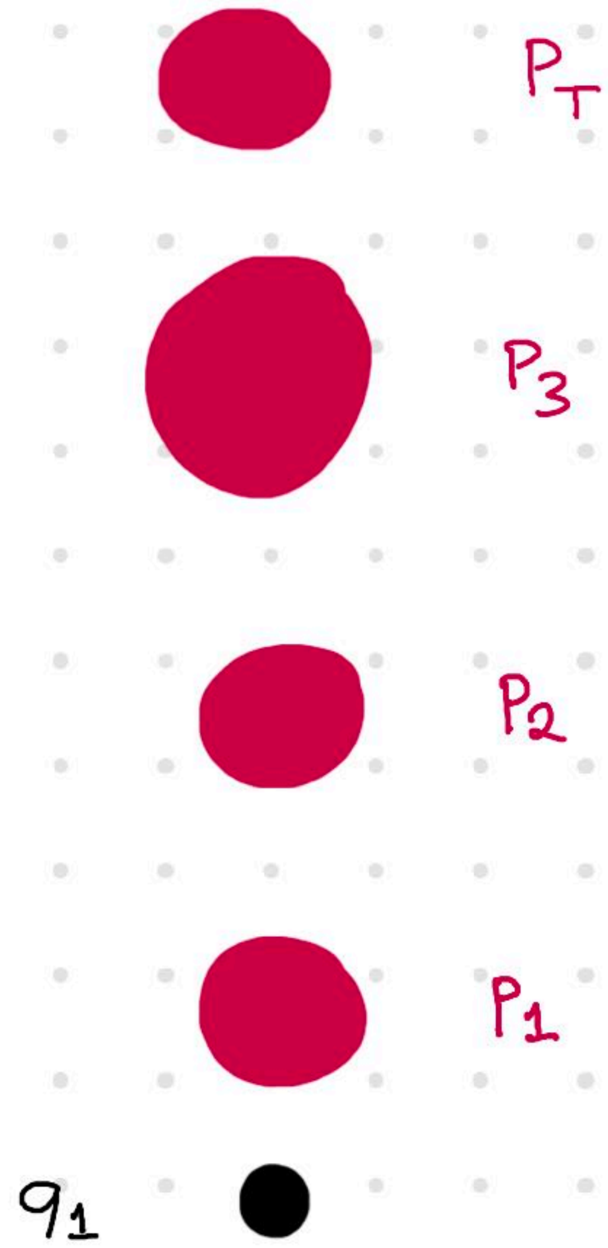

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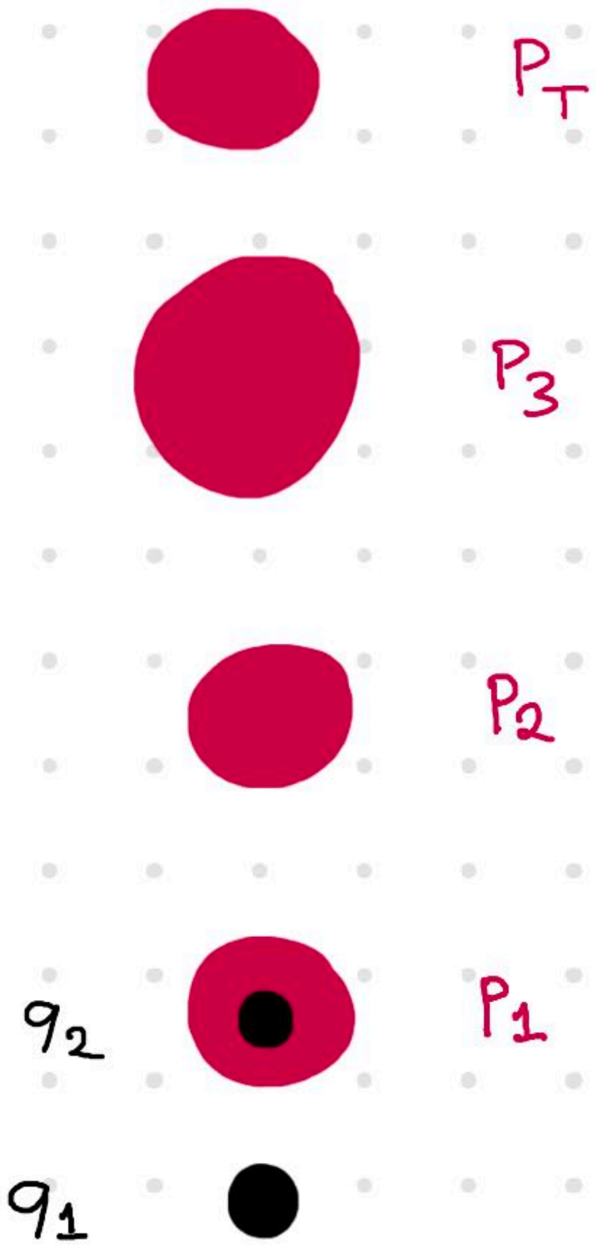

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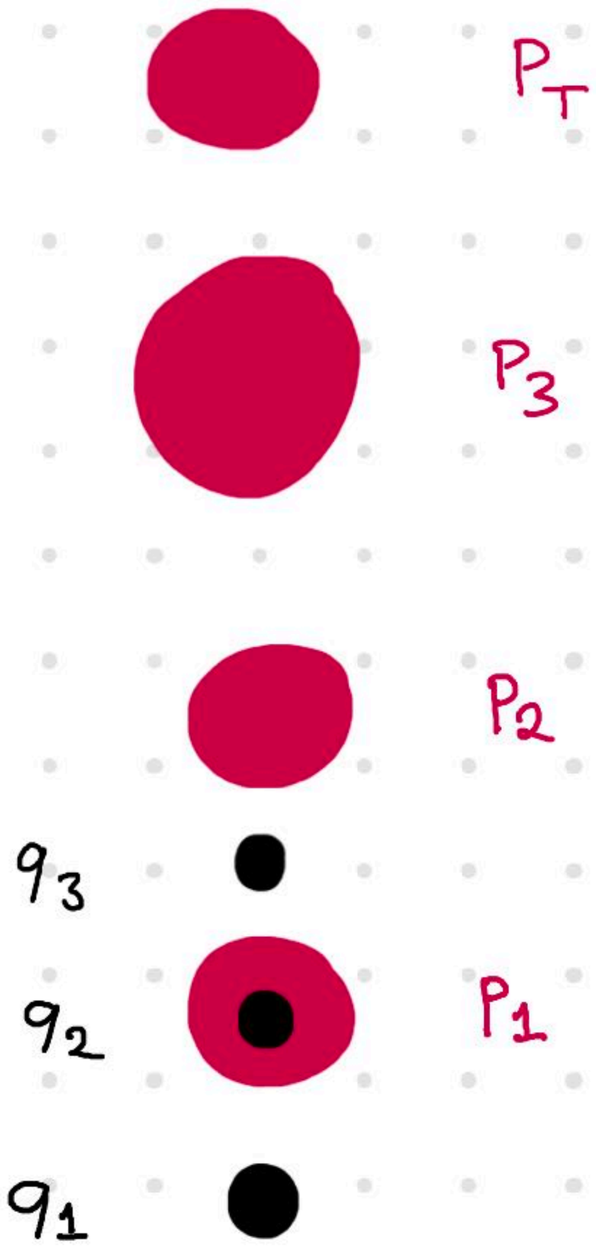

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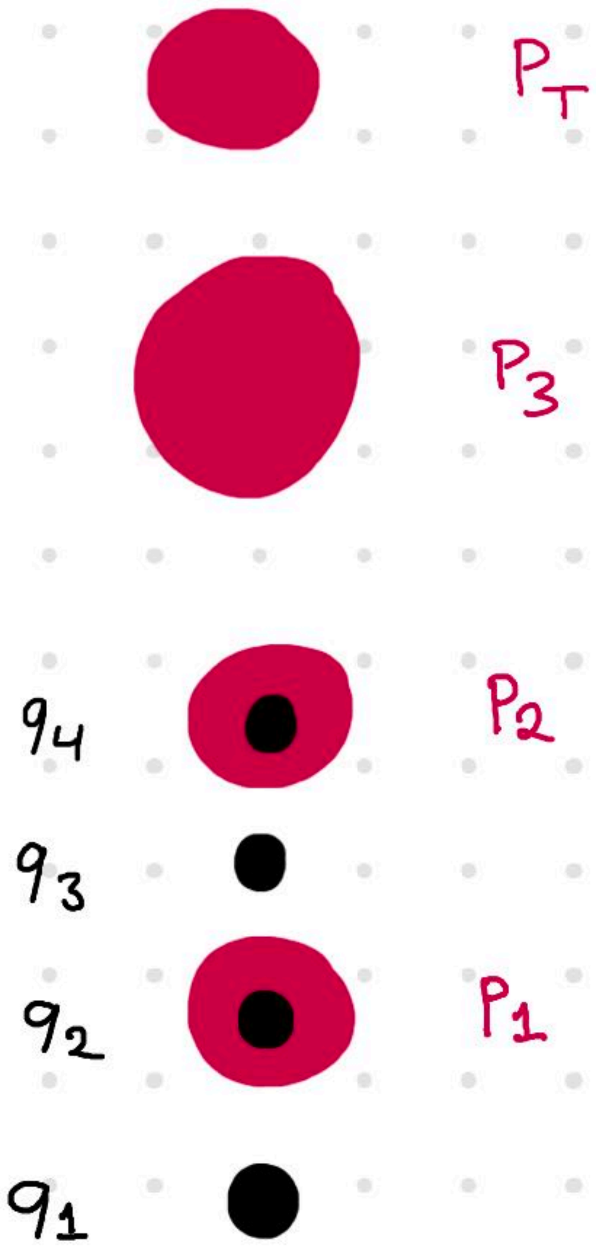

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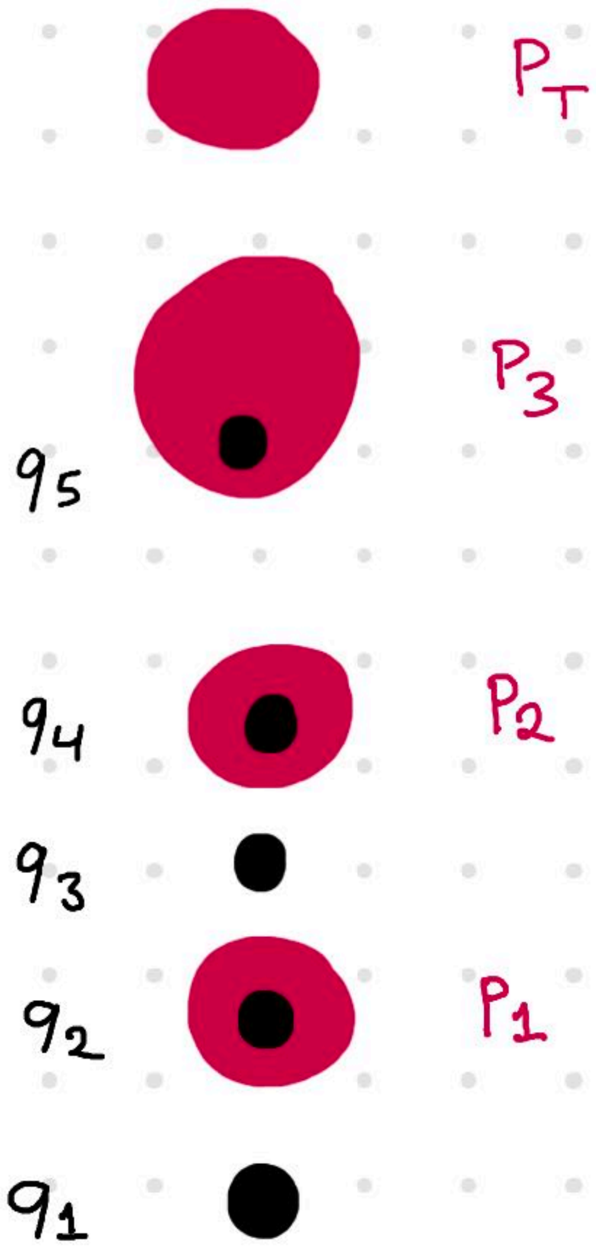

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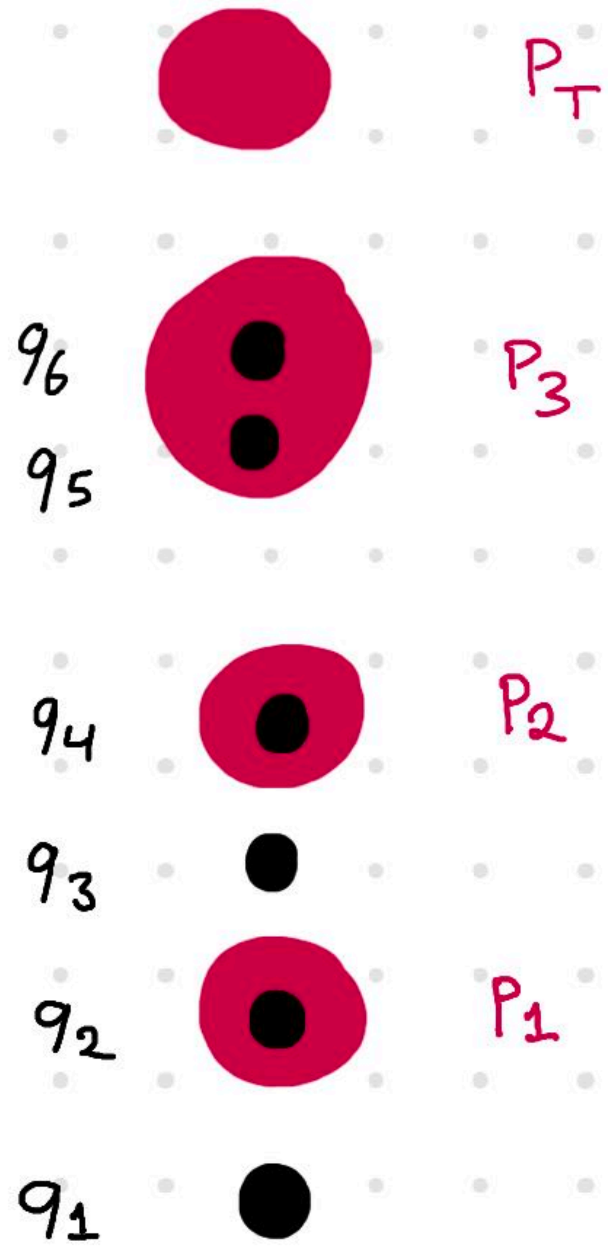

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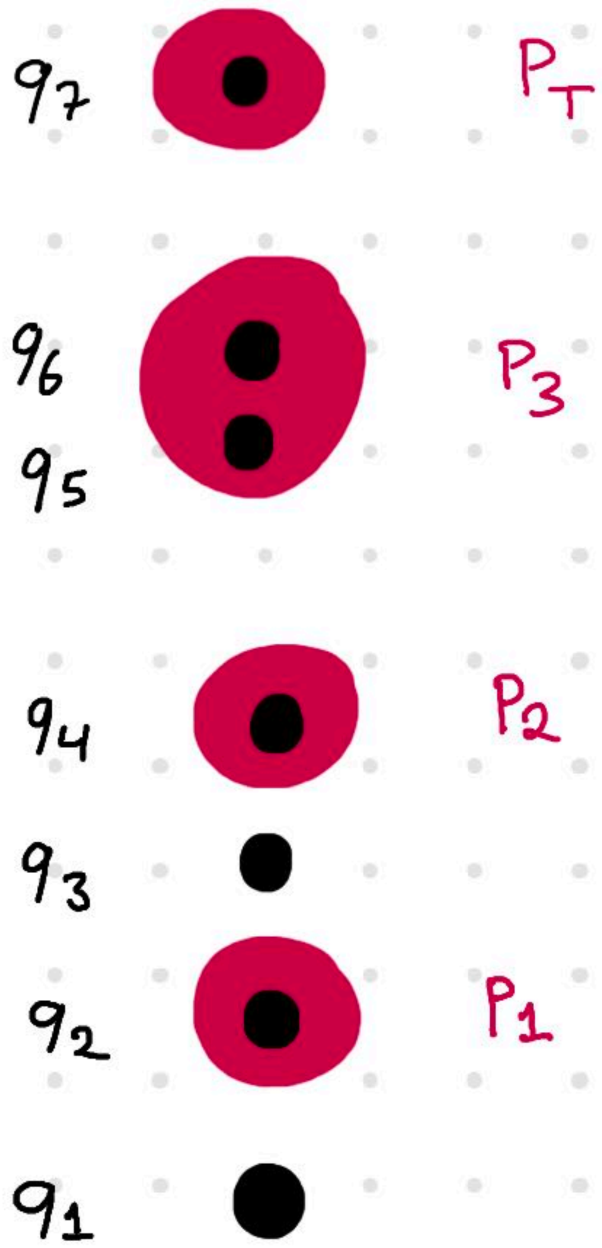

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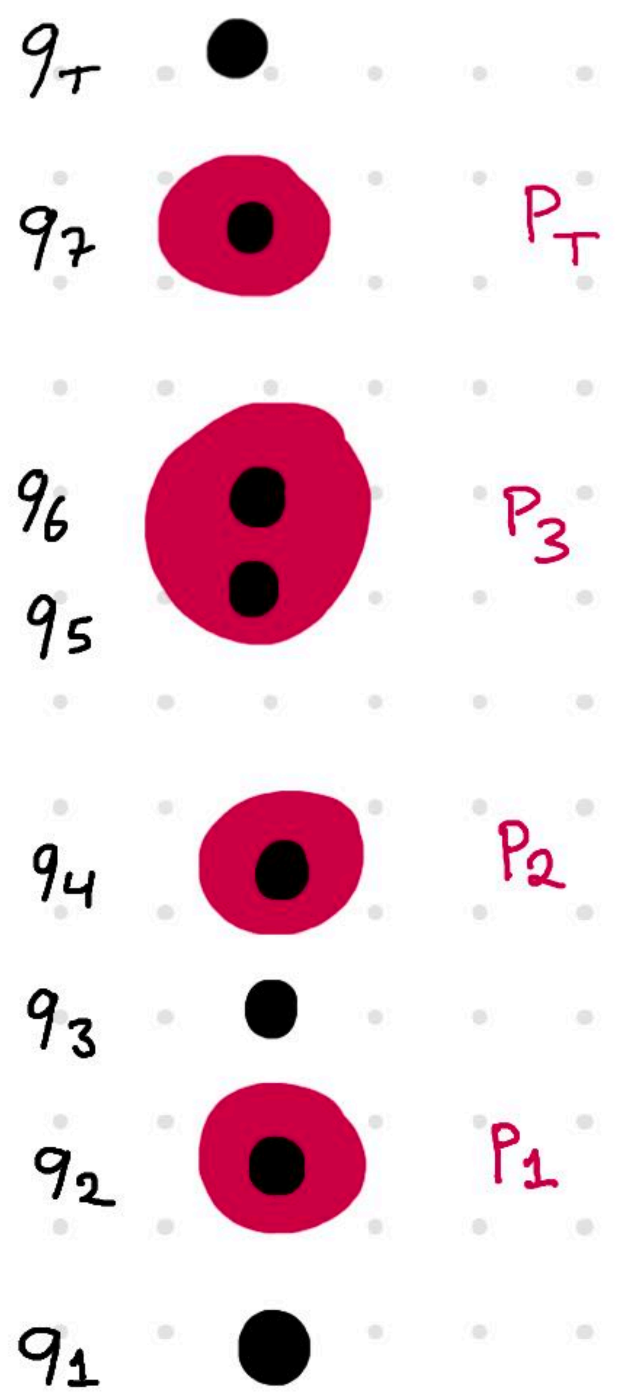
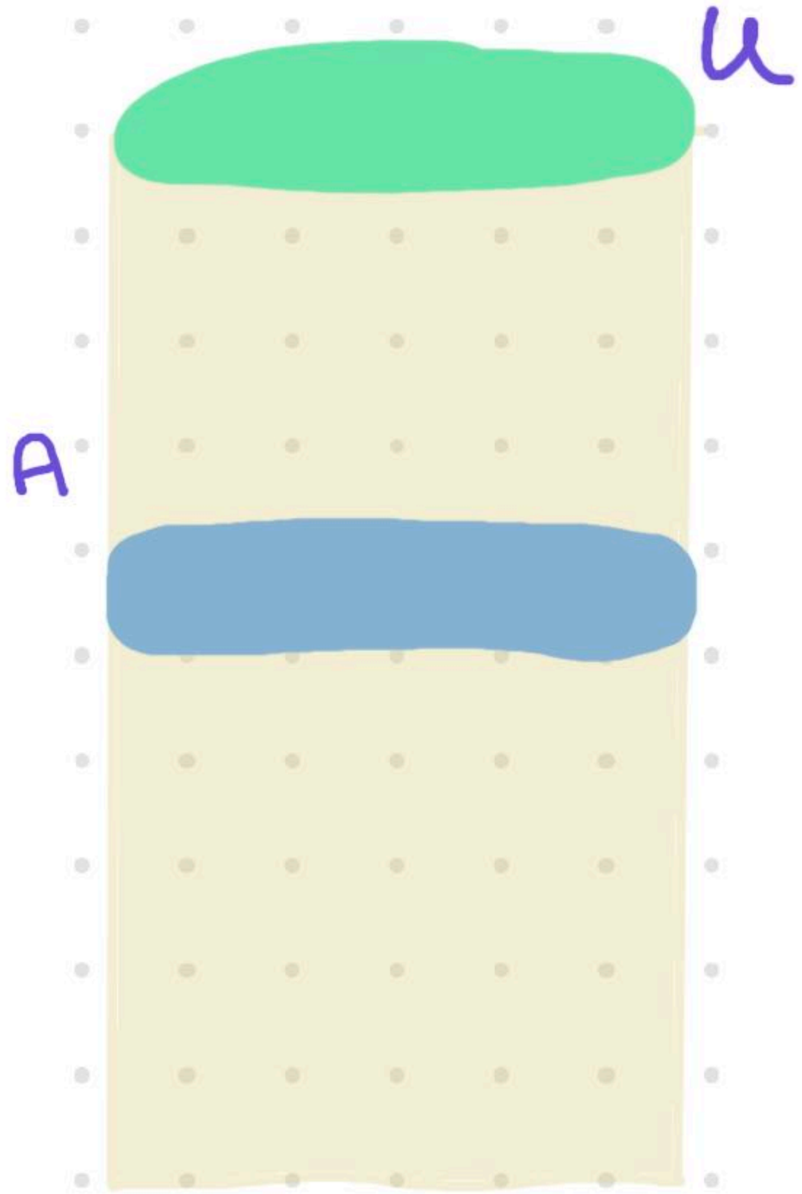


Coverage

$A \in \mathrm{Cov}^{-}(U)$

$\Updownarrow$ idea:

all paths landing in U refinable to A



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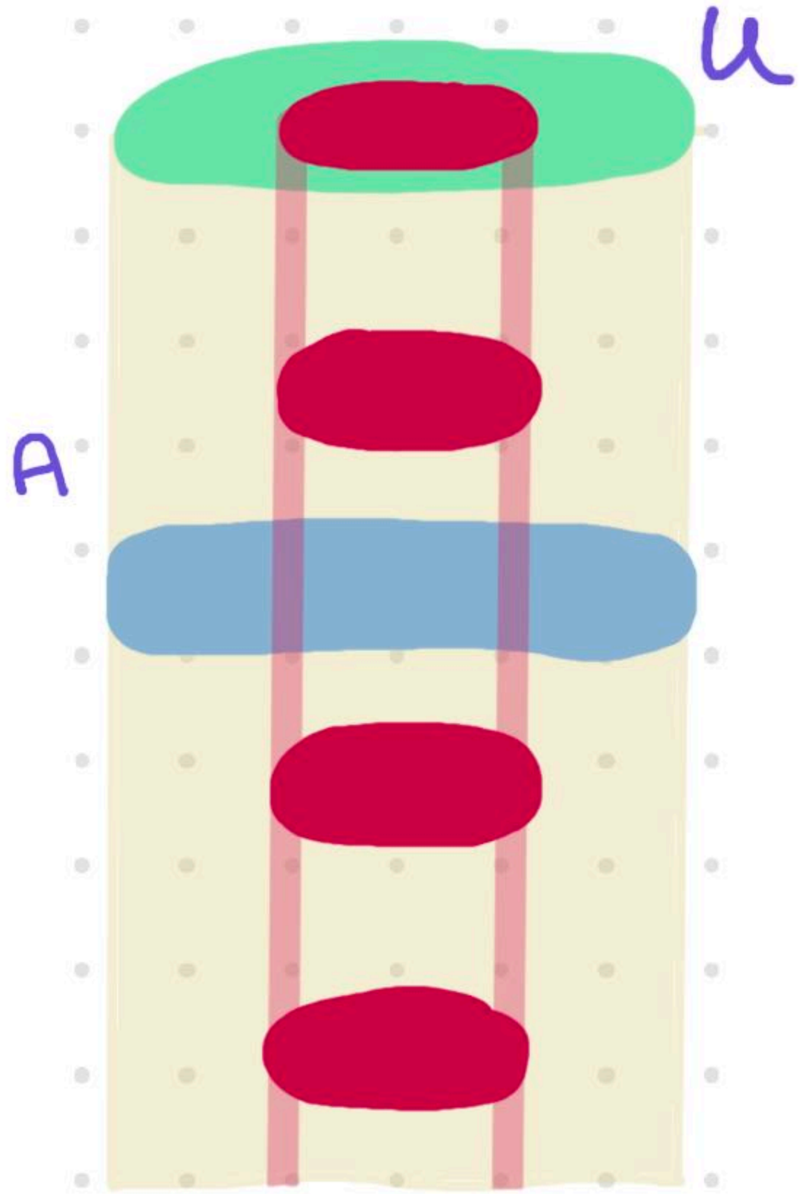
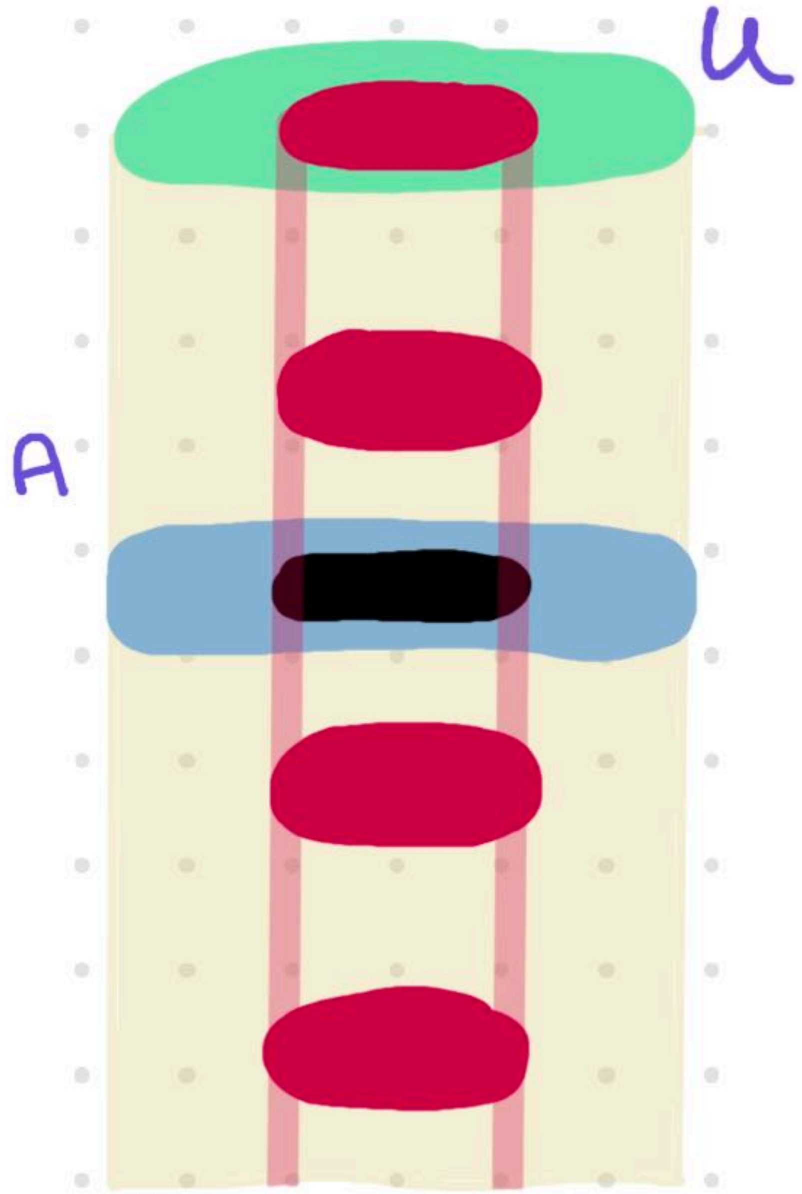


Coverage

$A \in \overline{Cov}(U)$

idea: $\Updownarrow$

all paths landing in U refinable to A



Coverage

$A \in \overline{\mathrm{Cov}}(U)$

idea: $\Updownarrow$

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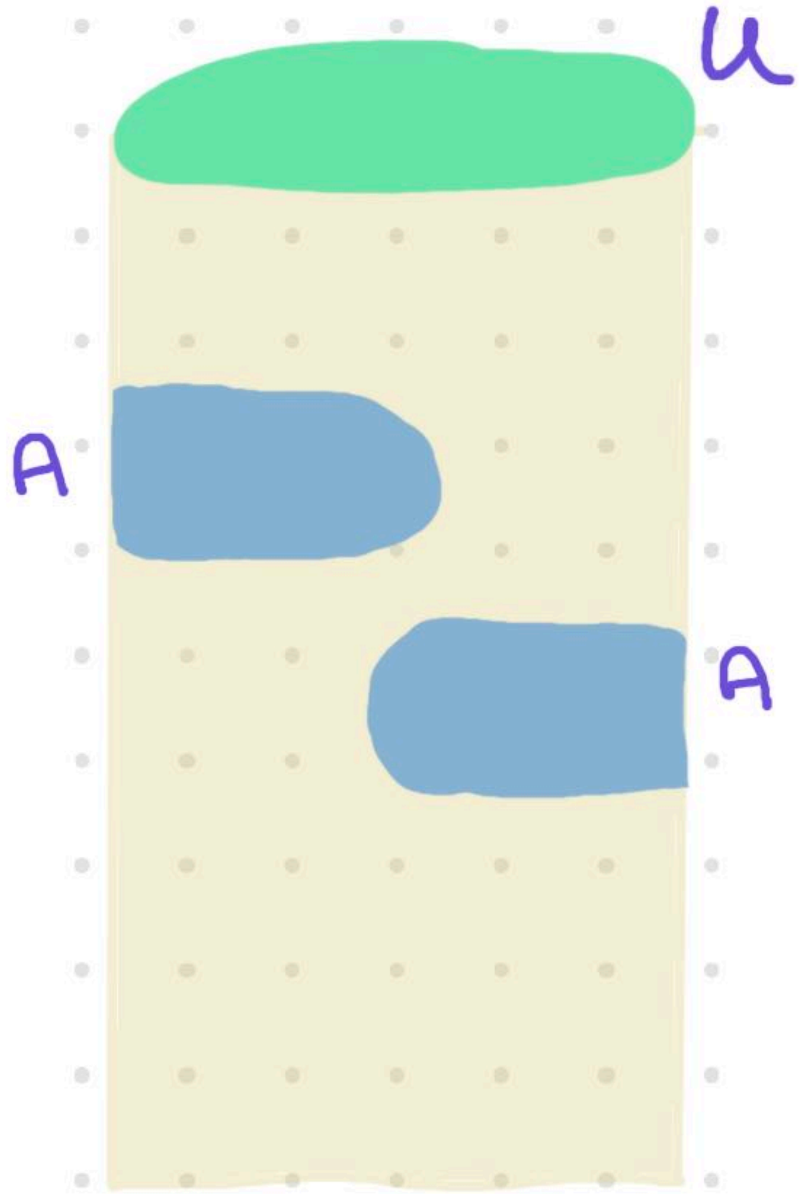

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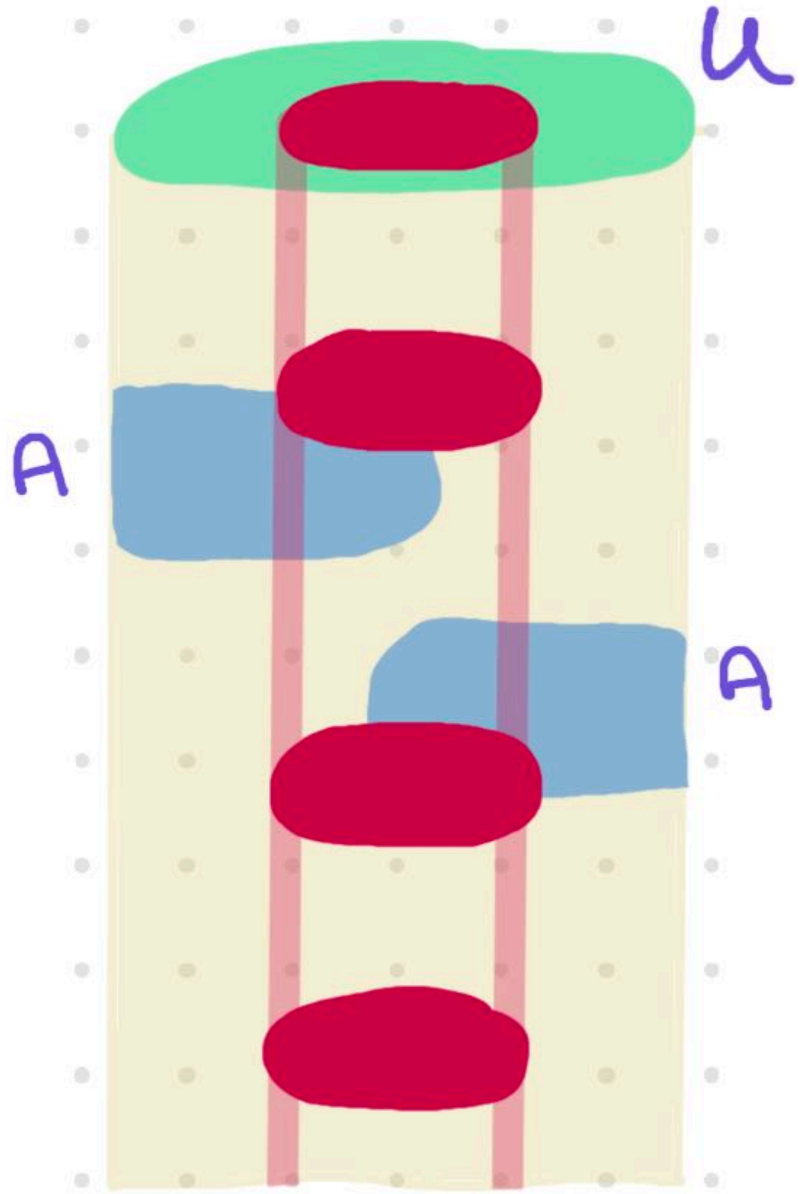

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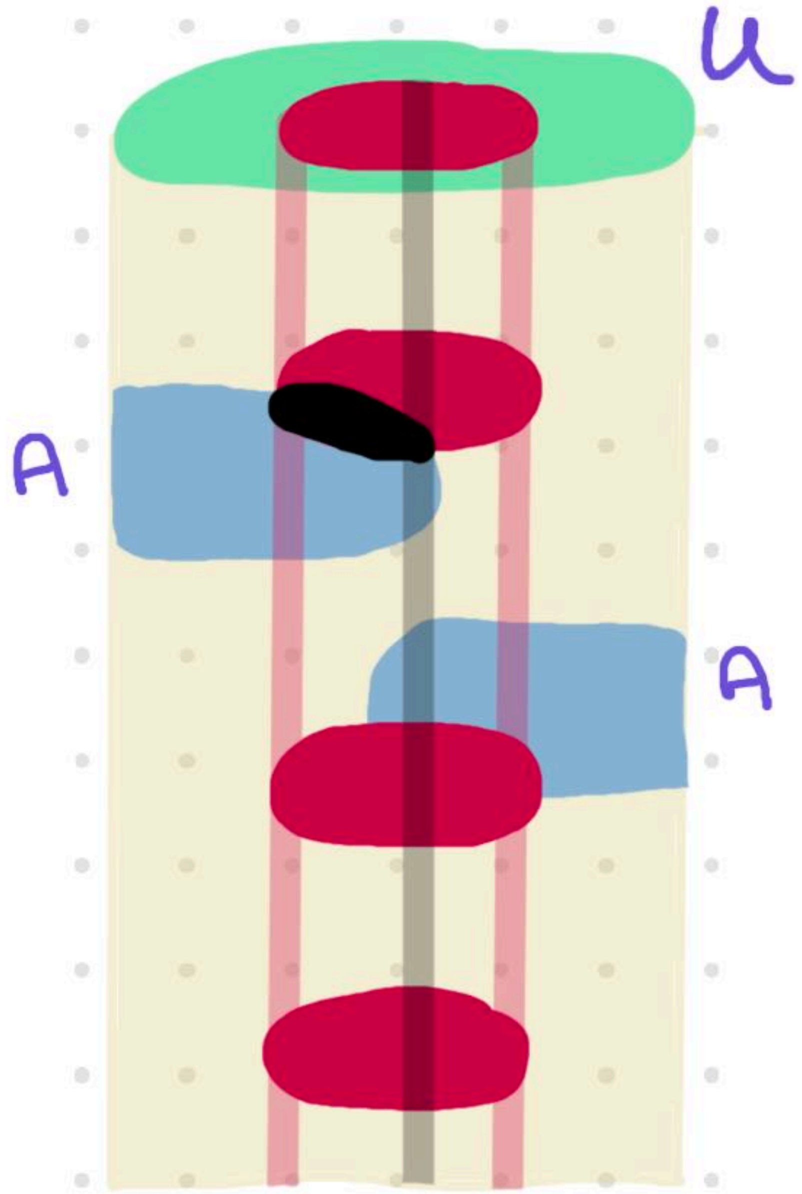

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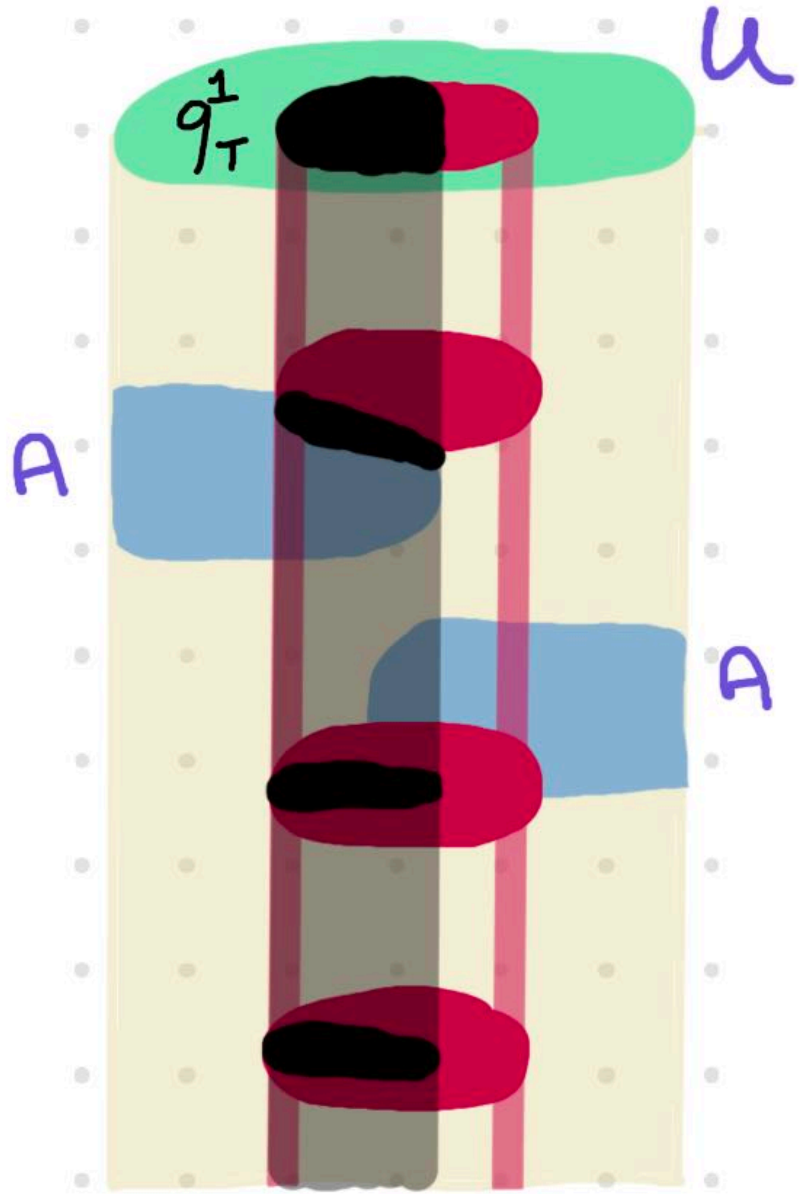

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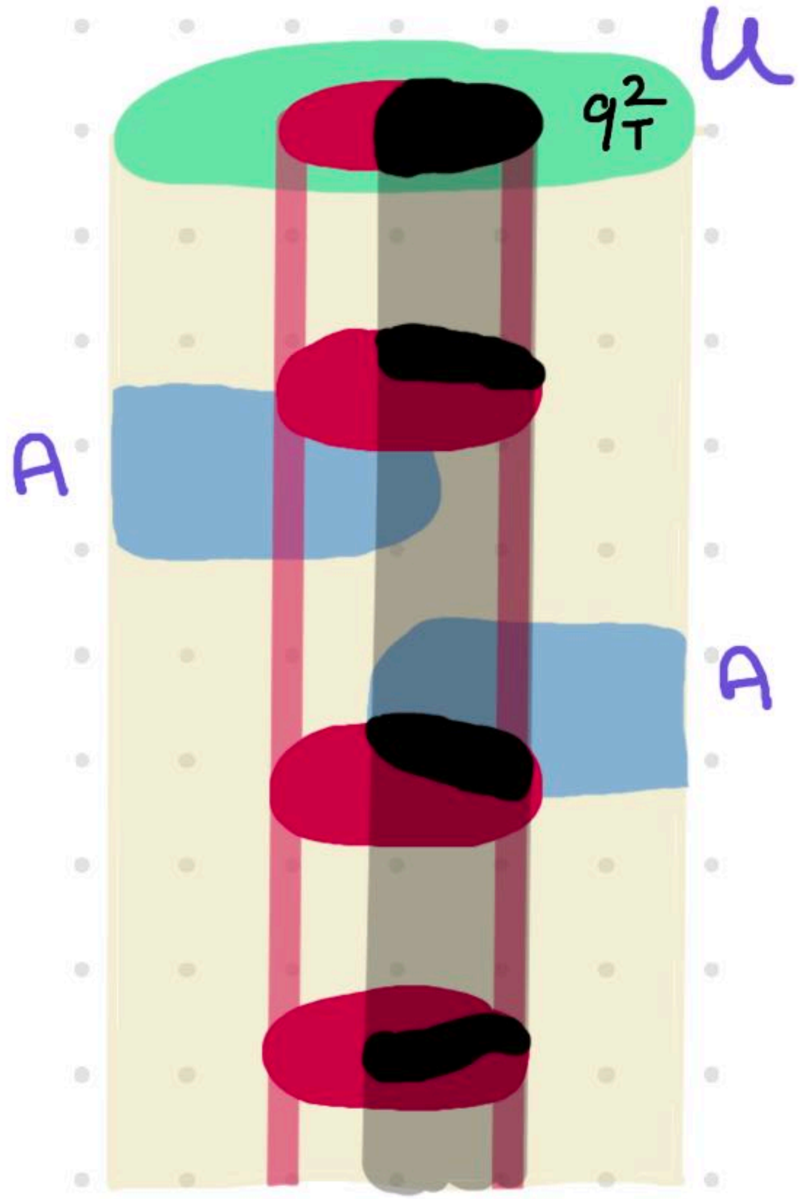


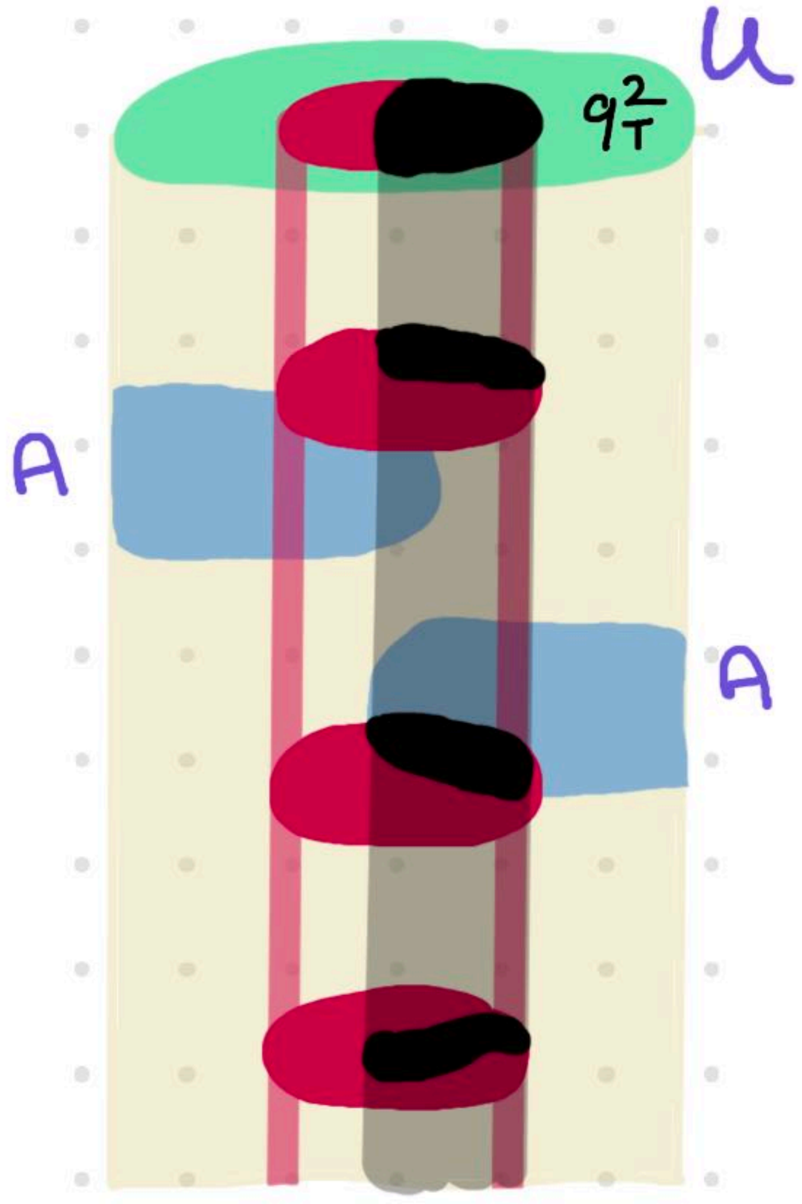
Coverage

$A \in \mathrm{Cov}^{-}(U)$

def: $\Updownarrow$

all paths landing in U ~~refinable~~ to A

Locally past refinable



Coverage

$A \in \mathrm{Cov}^{-}(U)$

def: $\Updownarrow$

all paths landing in U ~~refinable~~ to A

locally past refinable

Local past refinement:

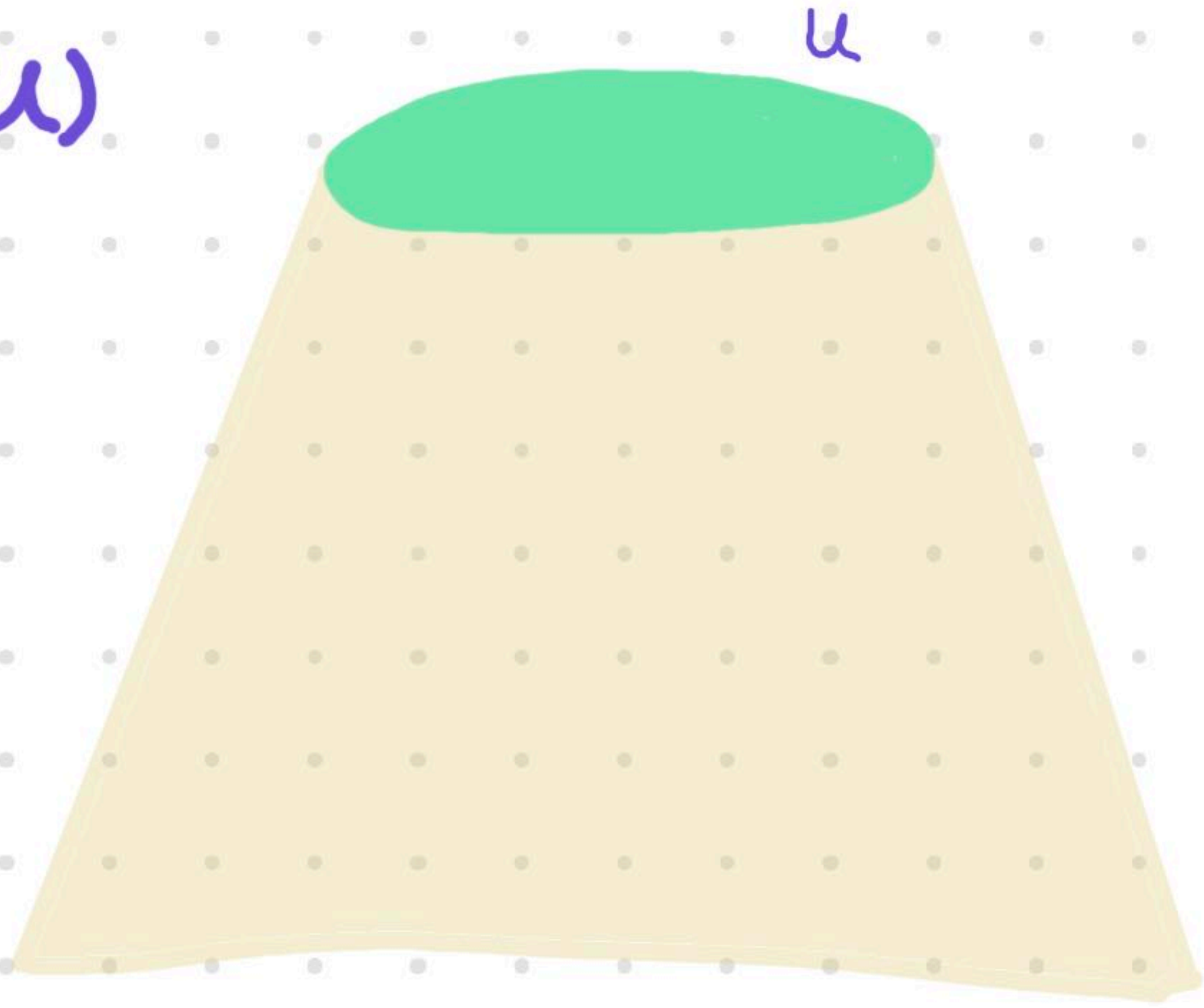
$q^i \Subset P|_{q^i_T}$

$\bigvee q^i_T = P_T$

Properties

identity

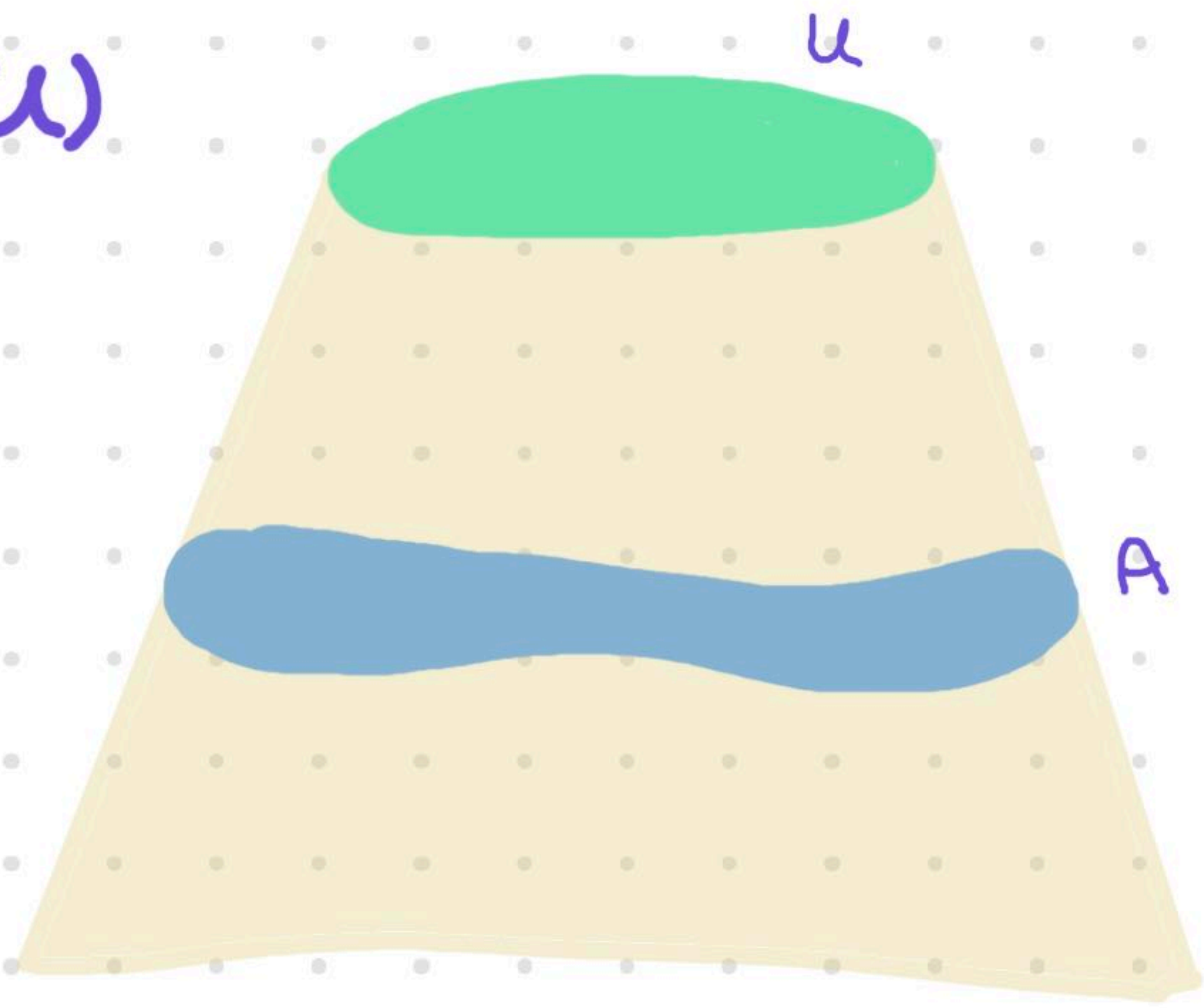
- $u \in \mathrm{Cov}^-(U)$, $\forall U \in \mathrm{Cov}^-(U)$



Properties

Pullback stability

- $\mathcal{U} \in \mathrm{Cov}^-(U)$, $\downarrow\mathcal{U} \in \mathrm{Cov}^-(U)$
- $A \in \mathrm{Cov}^-(U)$, $W \sqsubseteq U$
  $\Longrightarrow A \wedge \downarrow W \in \mathrm{Cov}^-(W)$



Properties

Pullback stability

- $U \in \mathrm{Cov}^-(U)$, $\downarrow U \in \mathrm{Cov}^-(U)$
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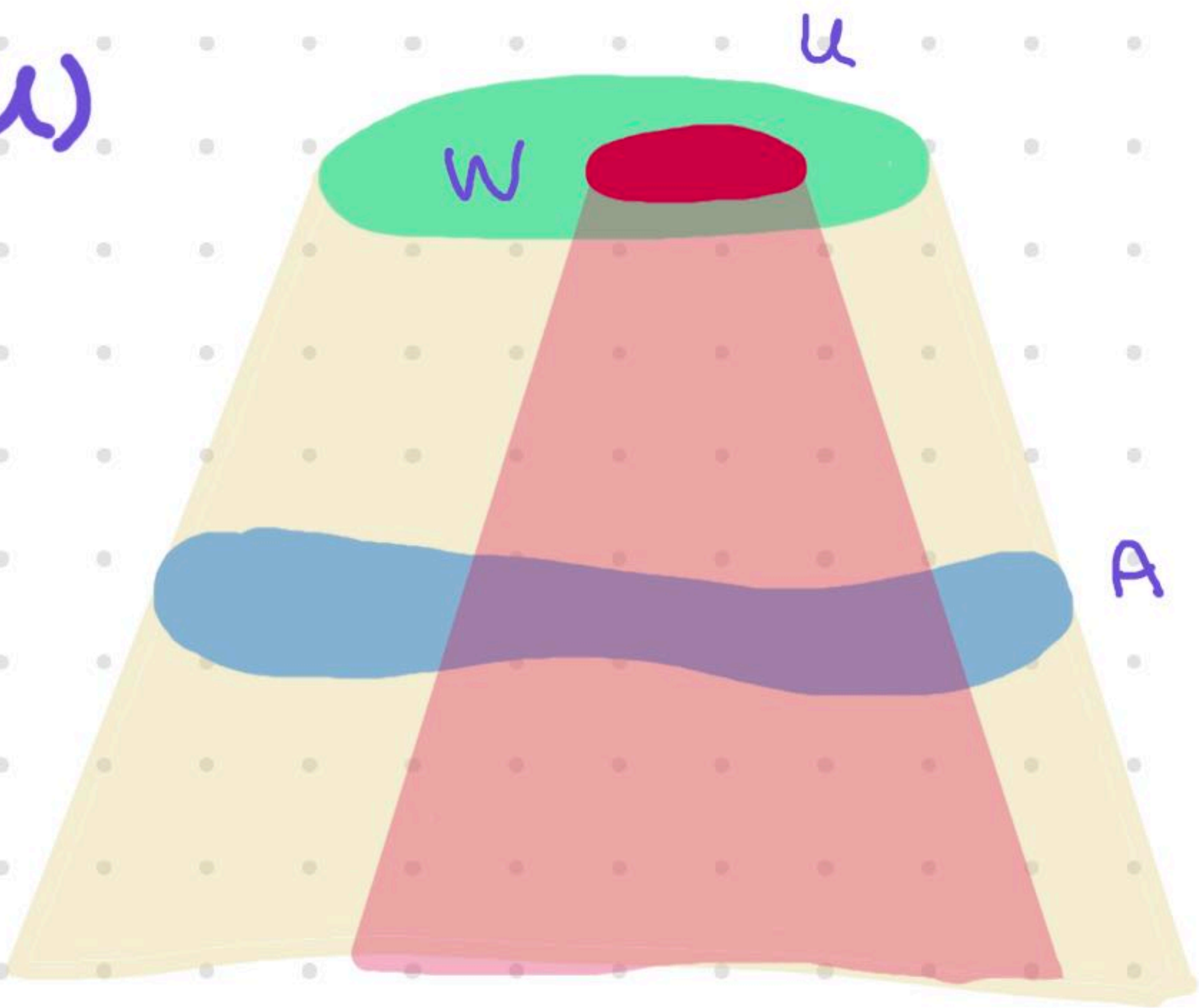

Properties

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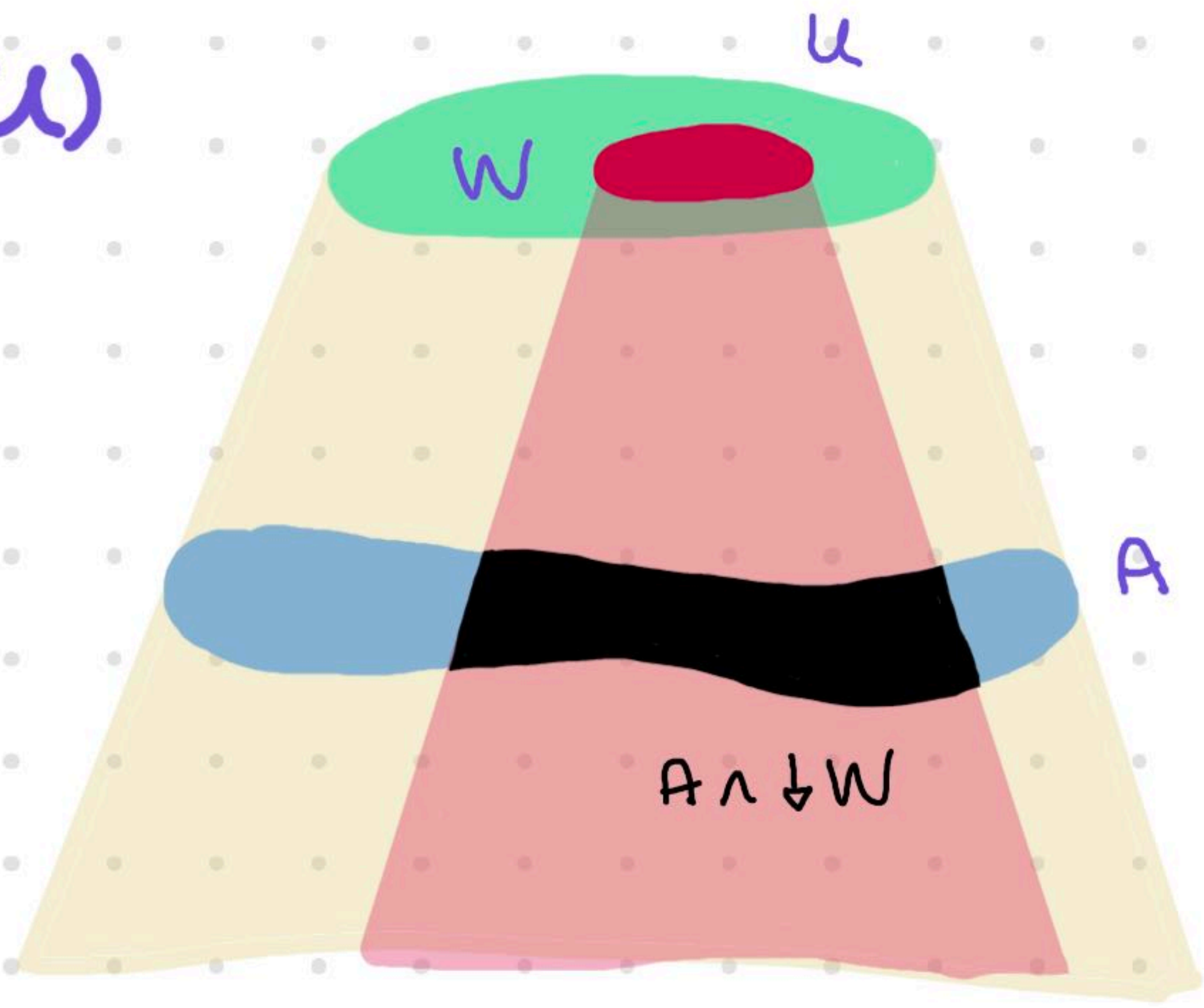
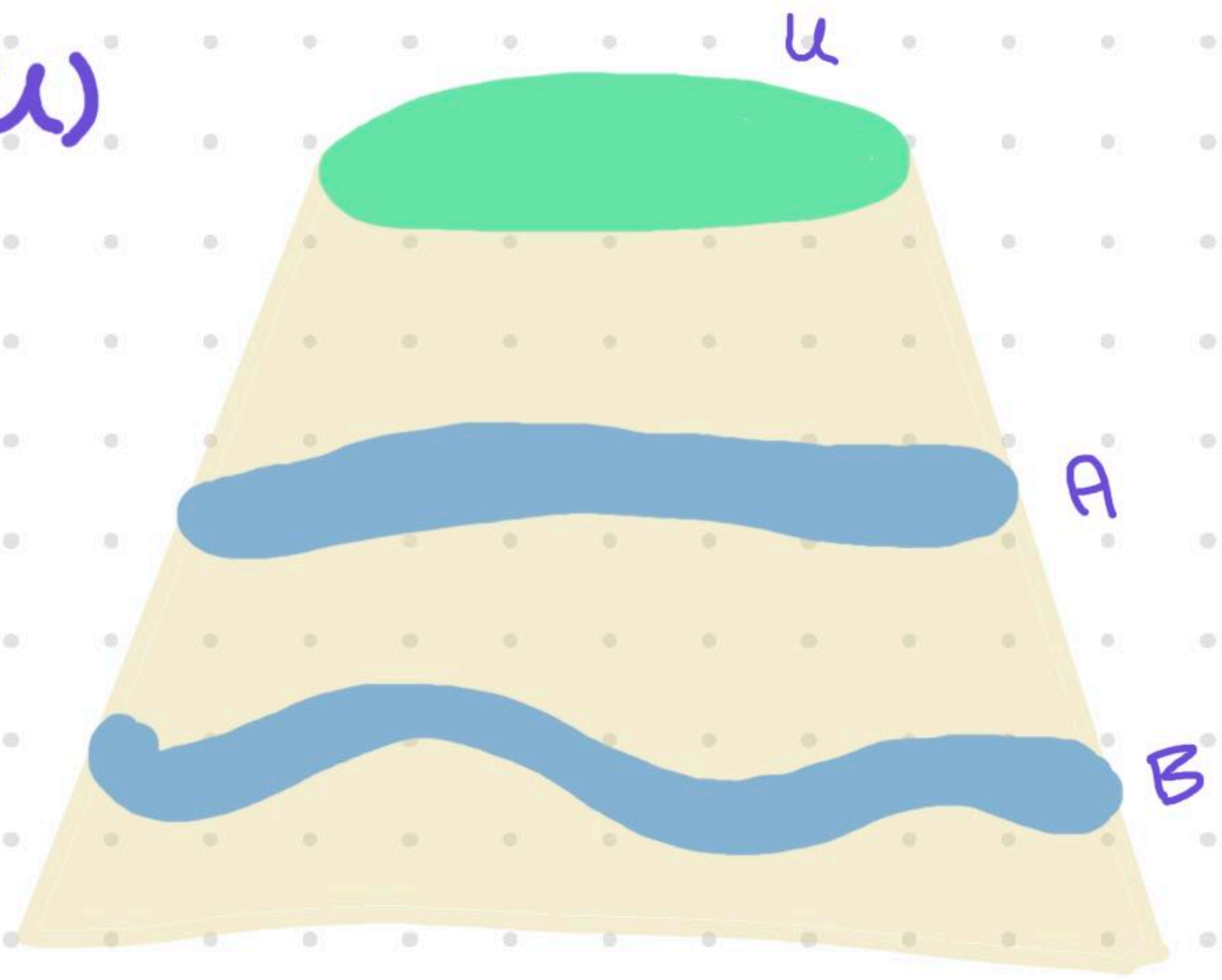


Properties

local characteristic

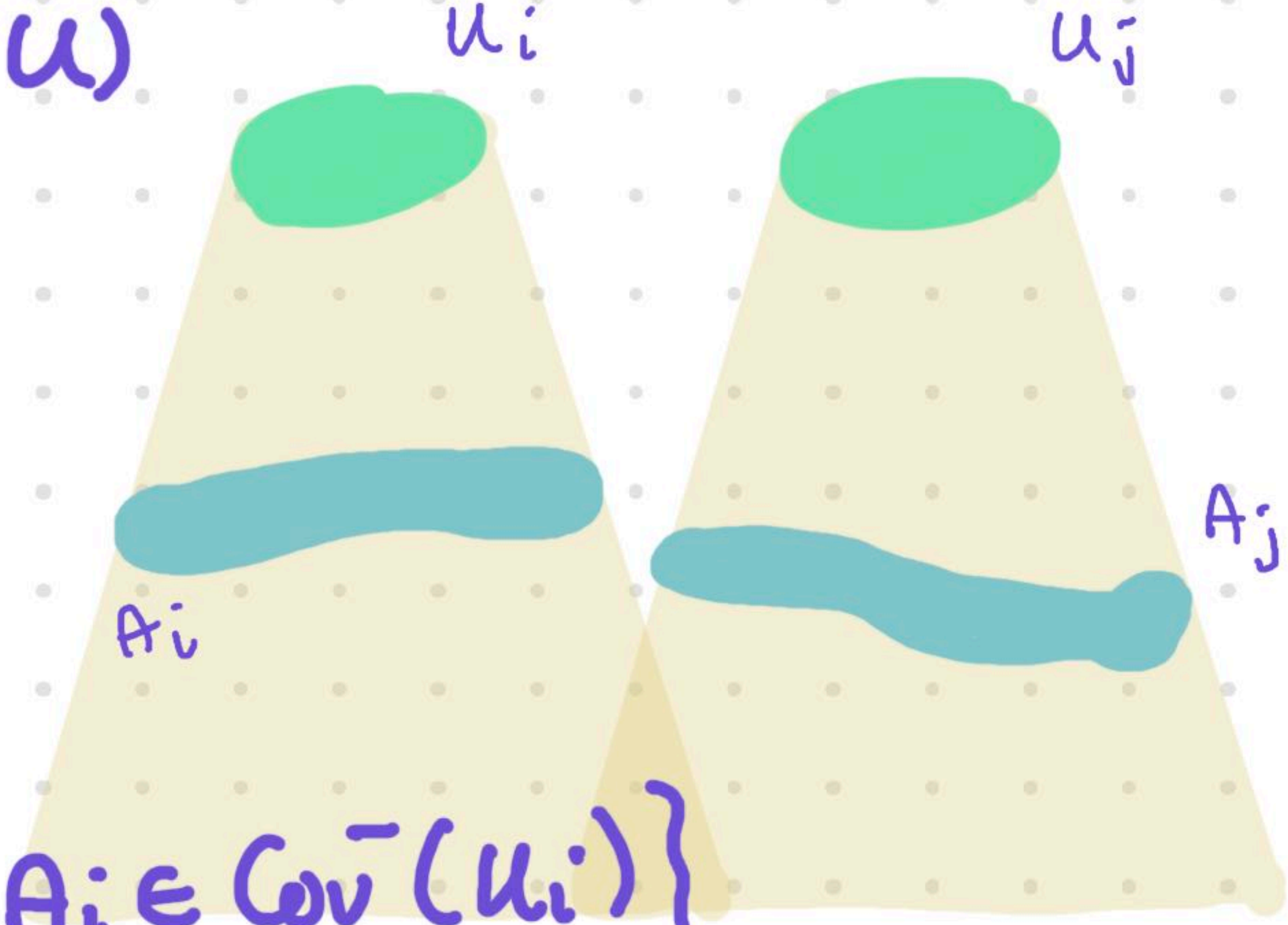
- $U \in \mathrm{Cov}^{-}(U)$, $\downarrow U \in \mathrm{Cov}^{-}(U)$
- $A \in \mathrm{Cov}^{-}(U)$, $W \sqsubseteq U$
  $\Longrightarrow A \wedge \downarrow W \in \mathrm{Cov}^{-}(W)$
- $B \in \mathrm{Cov}^{-}(A)$, $A \in \mathrm{Cov}^{-}(U)$
  $\Longrightarrow B \in \mathrm{Cov}^{-}(U)$



Properties

local characteristic

- $U \in Cov^-(U)$, $\downarrow U \in Cov^-(U)$
- $A \in Cov^-(U)$, $W \sqsubseteq U$
  $\implies A \wedge \downarrow W \in Cov^-(W)$
- $B \in Cov^-(A)$, $A \in Cov^-(U)$
  $\implies B \in Cov^-(U)$
- $Cov^-(\bigvee U_i) = \{\bigvee A_i : A_i \in Cov^-(U_i)\}$



Sheaves?

$R \in \bar{J}(U)$

sieve

$\Updownarrow$

$\bigvee R \in \overline{Cov}(U)$

Sheaves?

"T-Grothendieck topologies"

(monad

sieve $R \in \bar{J}(U)$

$$\Updownarrow$$

$$\forall R \in \overline{\mathrm{Cov}}(U)$$

Sheaves?

"T-Grothendieck topologies"

sieve $R \in J^{-}(U)$

$\Updownarrow$

$V R \in Co\bar{v}(U)$

(i) the maximal sieve $t_{T(C)}$ is in $J(C)$;

(ii) if $S \in J(C)$ then $T(h)^*(S) \in J(D)$ for any arrow $h\colon D \to C$;

(iii) if $S \in J(C)$ and R is a sieve on $T(C)$ such that $(\mu_C \circ T(h))^*(R) \in J(D)$ for all $h\colon D \to T(C)$ in S, then $R \in J(C)$.

Sheaves?

"T-Grothendieck topologies"

$R \in J^{-}(U)$

sieve

$\Updownarrow$

$V R \in Co\bar{v}(U)$

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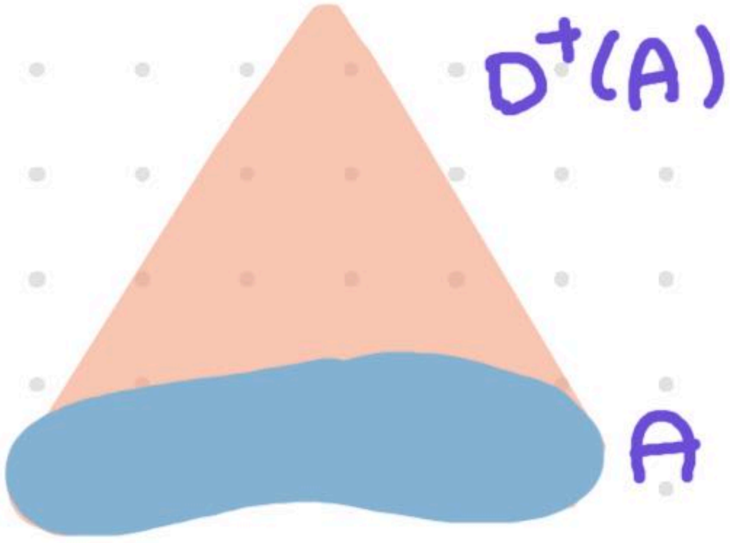
~~Def.~~/Thm. J^{-} is a $\downarrow$-Grothendieck topology

Sheaves?

$$D^+(A) := \bigvee \{ W : A \in \mathrm{Co}\bar{v}(W) \}$$

future domain of dependence



Sheaves?

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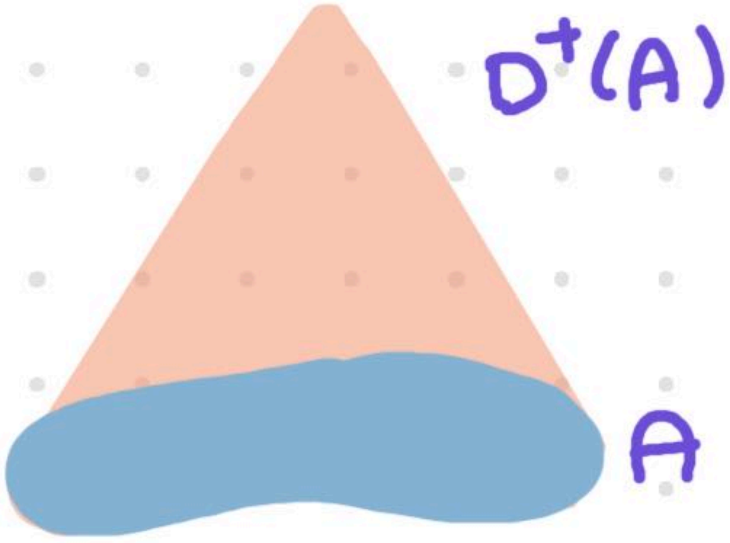

D^+-sheaf : compatible families on $R \in J(A)$ should glue uniquely to $D^+(A)$

e.g. sheaf of soln. to wave eqn. on Minkowski space

Sheaves?

- T-topologies
- D-sheaves

toposes?
internal logic?

- applications to spacetimes

"paradox" of hole-freeness

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- D-sheaves

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Thanks!