Countable and Dependent Choice and Countably Distributive Toposes

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Countable choice (CC):

$$\forall n \in \mathbb{N}\ \exists x \in X,\ P(n,x) \Rightarrow \exists f \in X^{\mathbb{N}}\ \forall n \in \mathbb{N},\ P(n, f(x))$$

Dependent choice (DC):

$$\forall x \in X\ \exists y \in X,\ R(x,y) \Rightarrow$$
$$\forall x \in X\ \exists f \in X^{\mathbb{N}},\ f(0) = x \wedge \forall n \in \mathbb{N},\ R(f(n), f(n+1))$$

- Not difficult to formulate in internal logic of elementary topos with NNO
- $AC \Rightarrow DC \Rightarrow CC$, but converse implications fail.

We begin with logic-free characterisations of CC and DC in elementary toposes with countable limits, which are probably folklore.

Detailed proofs can be found in the masters dissertation of my former student Severin Mejak

S. Mejak
Topos models of set-theoretic principles
Masters dissertation
Faculty of Mathematics and Physics, University of Ljubljana
2019

Logic-free characterisations of countable choice

Proposition T.f.a.e. for an elementary topos $\mathcal{E}$ with NNO

- $\mathcal{E} \vDash CC$
- The functor $(-)^{\mathbb{N}} : \mathcal{E} \to \mathcal{E}$ preserves epis | A standard formulation of CC in elementary toposes

Moreover if $\mathcal{E}$ has countable limits, another equivalent statement is

- If $X_1 \xrightarrow{e_1} Y_1$, $X_2 \xrightarrow{e_2} Y_2$, ... are epis then so is

$$\begin{array}{c} X_1 \times X_2 \times X_3 \times \cdots \\ \Big\downarrow e_1 \times e_2 \times e_3 \times \cdots \\ Y_1 \times Y_2 \times Y_3 \times \cdots \end{array}$$

i.e., the product functor

$$\Pi : \mathcal{E}^{\omega} \to \mathcal{E}$$

preserves epis

A logic-free characterisation of dependent choice

<u>Proposition</u> T.f.a.e. for an elementary topos $\mathcal{E}$ with countable limits

- $\mathcal{E} \models DC$
- Every ω^{op}-diagram of epis

$$\cdots \twoheadrightarrow \cdot \twoheadrightarrow \cdot \twoheadrightarrow \cdot \twoheadrightarrow \cdot$$

  has a cone of epis
- For every ω^{op}-diagram of epis, the limit cone is a cone of epis.

The *atomic (Grothendieck) topology* is defined on a small category $\mathbb{C}$ if and only if $\mathbb{C}$ is *coconfluent*: every cospan completes to a commuting square

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Atomic topology: a sieve S is a cover $\Leftrightarrow S \neq \emptyset$

The Grothendieck topos $Sh(\mathbb{C}, at)$ of *atomic sheaves* is boolean.

__Proposition__ A sufficient condition for $\mathrm{Sh}(\mathbb{C}, at)$ to validate CC is that $\mathbb{C}$ is __ω-coconfluent:__ every ω-cospan in $\mathbb{C}$

$$
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[dl] & \bullet \arrow[dll] & \bullet \arrow[dlll] & \cdots \\
\bullet & & & \cdots &
\end{tikzcd}
$$

has a cone.

(ω-coconfluent $\Leftrightarrow$ the atomic topology is closed under __countable__ intersections)

__Proposition__ A sufficient condition for $\mathrm{Sh}(\mathbb{C}, at)$ to validate DC is that every ω^{op}-chain in $\mathbb{C}$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

has a cone.

<u>Theorem</u> T.f.a.e. for a Grothendieck topos $\mathcal{E}$

1. $\mathcal{E} \equiv Sh(\mathbb{C}, J)$ where J is closed under countable intersections.

2. $\mathcal{E} \equiv Sh(\mathbb{C}, J)$ for which the associated sheaf functor $\underline{a} : PSh(\mathbb{C}) \longrightarrow Sh(\mathbb{C}, J)$ preserves countable limits.

3. $\mathcal{E}$ validates CC and satisfies: for all families $(J_i)_{i \in I}$ of sets and $(A_{i,j})_{i \in I, j \in J_i}$ of objects with I countable, the canonical map

$$\coprod_{f \in \prod_{i \in I} J_i} \prod_{i \in I} A_{i, f(i)} \longrightarrow \prod_{i \in I} \coprod_{j \in J_i} A_{i,j}$$

is an isomorphism.

We call a Grothendieck topos satisfying 3 a

<u>Countably distributive topos</u>

Examples of countably distributive toposes

- Presheaf toposes
- Grothendieck toposes with finite sites } Any such is $\equiv$ to a presheaf topos

<u>Examples of completely distributive</u> toposes

- Atomic toposes over ω-coconfluent categories
- Subtoposes of completely distributive toposes for which the inverse image of the geometric inclusion preserves countable limits

Do <u>countably geometric</u> morphisms (inverse image functors preserve countable limits) between countably distributive toposes provide a setting for studying classifying toposes for <u>countably geometric</u> theories (extending geometric logic with countable conjunctions and products) ?

<u>Conjecture</u> The topos of probability sheaves is the classifying countably distributive toposes for the countably geometric theory of non-separable measure algebras.

Outline proof of theorem

<u>1 ⇒ 2</u> It is standard to define $\underline{a} := (\cdot)^+ \circ (\cdot)^+$ where

$$(\cdot)^+ : \mathcal{E} \to \mathcal{E}$$

is Grothendieck's +-functor, which preserves finite limits.

If J is closed under countable intersections then it is easily verified that $\underline{a}$ preserves countable products.

$\underline{2 \Rightarrow 3}$ To verify the distributivity law in $Sh(\mathbb{C}, J)$,

suppose $(A_{i,j})_{i \in I, j \in J_i}$ is a family of sheaves

Writing $\coprod^s$ and $\coprod^p$ for sheaf and presheaf coproduct we have

$$\coprod^s_f \prod_i A_{i,f(i)} \cong \underline{a}\left(\coprod^p_f \prod_i A_{i,f(i)}\right) \cong \underline{a}\left(\prod_i \coprod^p_j A_{i,j}\right)$$

$$\cong \prod_i \left(\underline{a}\left(\coprod^p_j A_{i,j}\right)\right) \cong \prod_i \coprod^s_j A_{i,j}$$

$\underline{a}$ preserves countable products

To verify CC in $Sh(\mathbb{C}, J)$ we use a lemma.

<u>Lemma</u> If $A_1 \xrightarrow{d_1} B_1$, $A_2 \xrightarrow{d_2} B_2, \dots$ are J-dense monos in $PSh(\mathbb{C})$

then so is $\prod_n A_n \xrightarrow{\prod_n d_n} \prod_n B_n$.

<u>Proof of lemma</u> $\underline{a}\left(\prod_n d_n\right) \cong \prod_n \left(\underline{a}(d_n)\right)$ ($\underline{a}$ preserves countable products)

Since each d_n is dense, the r.h.s. is a countable product of isos

Hence the r.h.s. is an iso (CC in meta-theory)

Hence $\prod_n d_n$ is J-dense. $\square$

To verify CC in $Sh(\mathbb{C}, J)$, suppose $A_1 \overset{e_1}{\twoheadrightarrow} B_1$, $A_2 \overset{e_2}{\twoheadrightarrow} B_2, \dots$

are epis in $Sh(\mathbb{C}, J)$ we verify that $\prod_i A_i \xrightarrow{\prod_i e_i} \prod_i B_i$

is also epi in $Sh(\mathbb{C}, J)$

Let $A_n \overset{q_n}{\twoheadrightarrow} C_n \overset{m_n}{\rightarrowtail} B_n$ be the epi-mono factorisation of e_n in $PSh(\mathbb{C})$.

Then $\prod_i e_i = (\prod_i m_i) \circ (\prod_i q_i)$ is the e-m factorisation of $\prod_i e_i$ in $PSh(\mathbb{C})$.

Since each e_n is epi in $Sh(\mathbb{C}, J)$, each q_n is J-dense, hence $\prod_i q_i$ is J-dense by lemma

Since the mono part of the e-m-factorisation of $\prod_i e_i$ in $PSh(\mathbb{C})$ is J-dense

and $\prod_i e_i$ is a map between sheaves, it is epi in $Sh(\mathbb{C}, J)$.

<u>3 ⇒ 1</u> Suppose $\mathcal{E}$ is a countably distributive Grothendieck topos.

Let $\mathbb{C}$ be a small full subcategory of $\mathcal{E}$ containing a generating set for $\mathcal{E}$ and closed under countable limits in $\mathcal{E}$.

Let J be the <u>cover topology</u> on $\mathbb{C}$

$$\{A_j \xrightarrow{c_j} B\}_{j \in J} \text{ is covering } \Leftrightarrow \{c_j\}_{j \in J} \text{ is jointly epi in } \mathcal{E}$$

By Giraud theorem / comparison lemma machinery $\mathcal{E} \simeq Sh(\mathbb{C}, J)$.

We show that J is closed under countable intersections.

Let $S_1, S_2, \ldots$ be covering sieves for an object B.

Each $S_n = \{A_{n,j} \xrightarrow{b_{n,j}} C\}_{j \in J_n}$ gives an epi $\coprod_{j \in J_n} A_{n,j} \overset{c_n := (b_{n,j})_j}{\twoheadrightarrow} C$

We have:

$$\begin{tikzcd}
\coprod_{f \in \prod_{i \in \mathbb{N}} J_i} P_f \arrow[rr] \arrow[dddd, two heads, "(e_f)_f"'] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \coprod_{f \in \prod_{i \in \mathbb{N}} J_i} \prod_{i \in \mathbb{N}} A_{i,f(i)} \arrow[dd, "\cong"] \arrow[dddd, bend right=60, "(d_f)_f :="'] & \text{(iso by distributivity)} \\
& & & \\
& & \prod_{i \in \mathbb{N}} \coprod_{j \in J_i} A_{i,j} \arrow[dd, two heads, "\prod_{i \in \mathbb{N}} c_i"] & \text{(epi by CC)} \\
& & & \\
C \arrow[rr, "\Delta"] & & \prod_{i \in \mathbb{N}} C &
\end{tikzcd}$$

where each $P_f \xrightarrow{e_f} C$ is given by the pullback in $\mathbb{C}$

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d, "e_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{i \in \mathbb{N}} A_{i,f(i)} \arrow[d, "d_f"] \\
C \arrow[r, "\Delta"] & \prod_{i \in \mathbb{N}} C
\end{tikzcd}$$

The left edge of the top diagram is epi (since pullback of epi) hence defines a cover $\{P_f \xrightarrow{e_f} C\}_{f \in \prod_{i \in \mathbb{N}} J_i}$, which by construction is contained in $\bigcap_{i \in \mathbb{N}} S_n$. By monotonicity of covering property $\bigcap_{i \in \mathbb{N}} S_i \in J$. $\square$