

Finitely accessible arboreal adjunctions and Hintikka formulae

Luca Reggio & Colin Riba

Department of Mathematics, Università degli Studi di Milano, Italy
LIP, ENS de Lyon, France

PSSL 109 (November 2024)

A reformulation of an old example (1/2)

Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (1/2)

Situation

$$\langle \bar{x} \mid \varphi \rangle$$

where

- ▶ $\bar{x} = x_1, \dots, x_n$
- ▶ φ finite conjunction of constraints $((x_i < x_j) \text{ or } (x_i = x_j))$

Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (1/2)

Situation

$$\langle \bar{X} \mid \varphi \rangle \xrightarrow{m} (M, <_M)$$

where

- ▶ $\bar{X} = x_1, \dots, x_n$
- ▶ φ finite conjunction of constraints $((x_i < x_j) \text{ or } (x_i = x_j))$
- ▶ m is an order embedding:

$$\begin{aligned} m(x_i) <_M m(x_j) &\iff (x_i < x_j) \text{ in } \varphi \\ m(x_i) = m(x_j) &\iff (x_i = x_j) \text{ in } \varphi \end{aligned}$$

Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.
(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

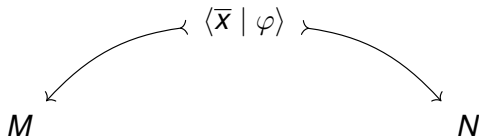
Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.
(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



Formulation inspired from Abramsky & Shah (2018, 2021).

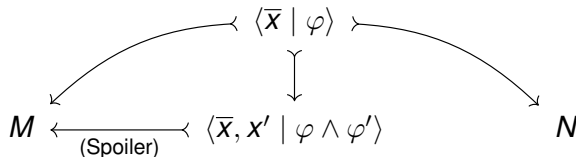
A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



Formulation inspired from Abramsky & Shah (2018, 2021).

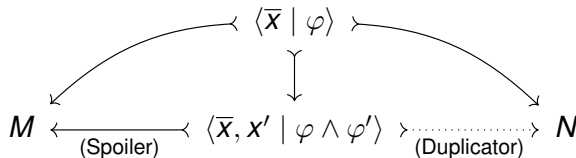
A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



Formulation inspired from Abramsky & Shah (2018, 2021).

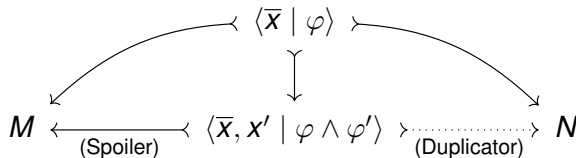
A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



or symmetrically.

Formulation inspired from Abramsky & Shah (2018, 2021).

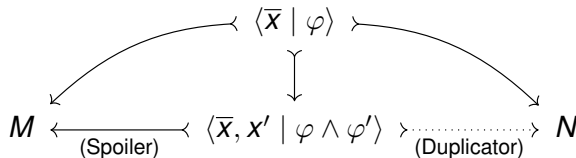
A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



or symmetrically.

- Duplicator wins since they can always respond.

Formulation inspired from Abramsky & Shah (2018, 2021).

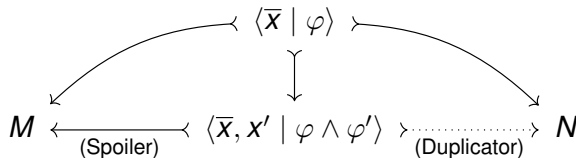
A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)



or symmetrically.

- Duplicator wins since they can always respond.

Corollary (... , Karp (1965))

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_\infty(<)$.

Formulation inspired from Abramsky & Shah (2018, 2021).

Toward game comonads

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

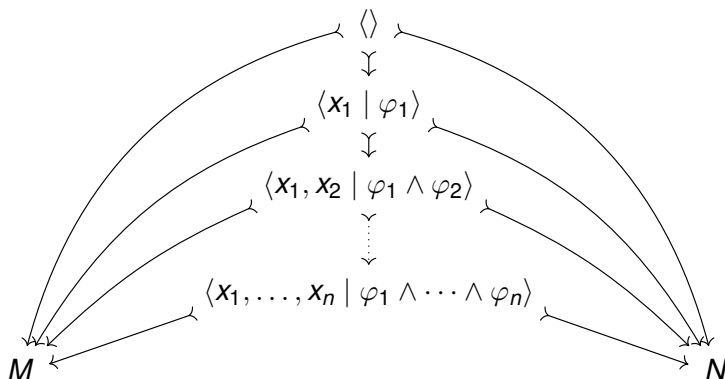
Toward game comonads: turn plays into structures

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

► Play

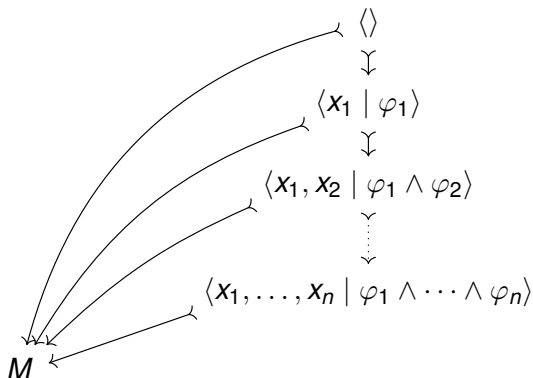


Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on M

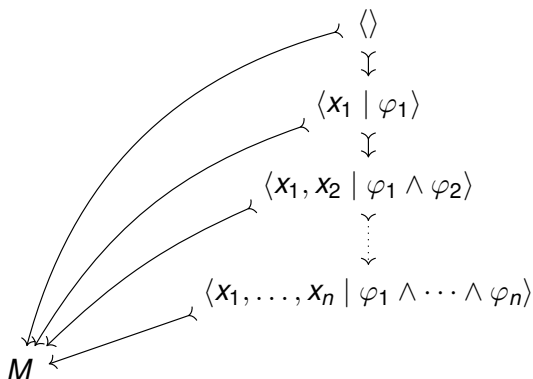


Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on M



is an element of a structure $R_{\text{EF}}(M)$ with carrier M^* .

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- ▶ Play projected on M is an element of a structure $R_{\text{EF}}(M)$ with carrier M^* .

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021), Abramsky & Marsden (2021, 2022)...

Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- ▶ Play projected on M is an element of a structure $R_{\text{EF}}(M)$ with carrier M^* .

Other examples

- ▶ Pebble games.
- ▶ Modal fragment, Hybrid fragment, Guarded fragments, ...

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021), Abramsky & Marsden (2021, 2022)...

Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

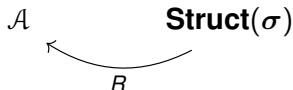
- ▶ Play projected on M is an element of a structure $R_{\text{EF}}(M)$ with carrier M^* .

Other examples

- ▶ Pebble games.
- ▶ Modal fragment, Hybrid fragment, Guarded fragments, ...

Adjunctions

- ▶ The $R(M)$ are structures with a tree order.



Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021), Abramsky & Marsden (2021, 2022)...

Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

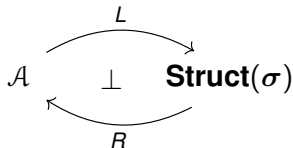
- ▶ Play projected on M is an element of a structure $R_{\text{EF}}(M)$ with carrier M^* .

Other examples

- ▶ Pebble games.
- ▶ Modal fragment, Hybrid fragment, Guarded fragments, ...

Adjunctions

- ▶ The $R(M)$ are structures with a tree order.
- ▶ In each case, R is a right adjoint.
- ▶ Comonads on **Struct**(σ).

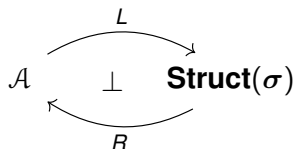


Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021), Abramsky & Marsden (2021, 2022)...

Arboreal categories

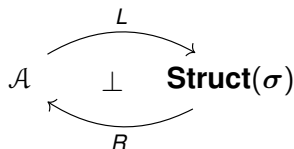
Abramsky & Reggio (2021, 2023).

Arboreal categories: motivations



Abramsky & Reggio (2021, 2023).

Arboreal categories: motivations



Conditions on $\mathcal{A}$ which yield well-behaved games.

Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

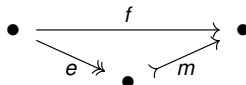
Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
each morphism f factors as

$$(e \in \mathcal{Q}, m \in \mathcal{M})$$



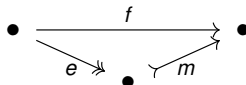
Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
each morphism f factors as

$(e \in \mathcal{Q}, m \in \mathcal{M})$



- Typically, the “embeddings” $m \in \mathcal{M}$ are embeddings of structures which are tree morphisms.

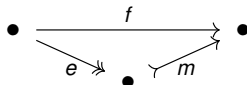
Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

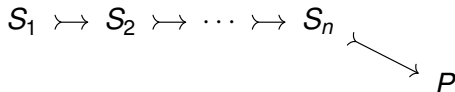
Arboreal category $\mathcal{A}$.

- ▶ Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
each morphism f factors as

$$(e \in \mathcal{Q}, m \in \mathcal{M})$$



- ▶ Typically, the “embeddings” $m \in \mathcal{M}$ are embeddings of structures which are tree morphisms.
- ▶ $P \in \mathcal{A}$ is a **path** when its $\mathcal{M}$ -subobjects form a finite chain



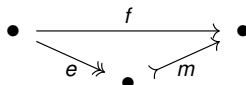
Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

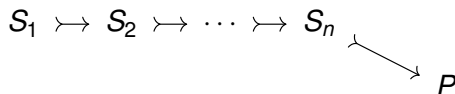
Arboreal category $\mathcal{A}$.

- ▶ Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
each morphism f factors as

$$(e \in \mathcal{Q}, m \in \mathcal{M})$$



- ▶ Typically, the “embeddings” $m \in \mathcal{M}$ are embeddings of structures which are tree morphisms.
- ▶ $P \in \mathcal{A}$ is a **path** when its $\mathcal{M}$ -subobjects form a finite chain



- ▶ Induced functor $\mathcal{A} \rightarrow \mathbf{Tree}$.

Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$(X, Y \in \mathcal{A})$

Abramsky & Reggio (2021, 2023).

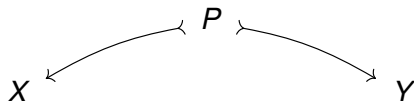
Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$(X, Y \in \mathcal{A})$

► Positions are spans of “embeddings”

$(P \text{ path})$



Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

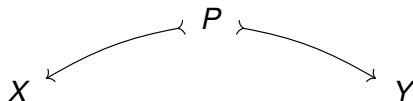
$(X, Y \in \mathcal{A})$

► Positions are spans of “embeddings”

$(P \text{ path})$

► Moves:

(played by Spoiler and Duplicator)



Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

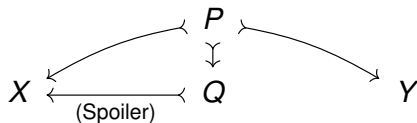
$(X, Y \in \mathcal{A})$

► Positions are spans of “embeddings”

$(P \text{ path})$

► Moves:

(played by Spoiler and Duplicator)



Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

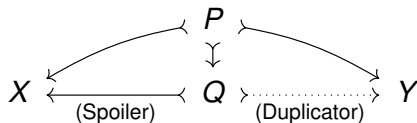
$(X, Y \in \mathcal{A})$

► Positions are spans of “embeddings”

$(P \text{ path})$

► Moves:

(played by Spoiler and Duplicator)



Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

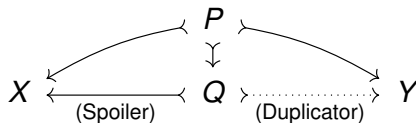
$(X, Y \in \mathcal{A})$

► Positions are spans of “embeddings”

$(P \text{ path})$

► Moves:

(played by Spoiler and Duplicator)



or symmetrically.

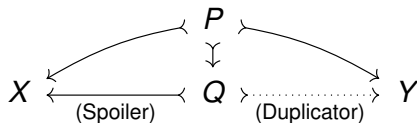
Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$(X, Y \in \mathcal{A})$

- Positions are spans of “embeddings” (P path)
- Moves: (played by Spoiler and Duplicator)



or symmetrically.

- Duplicator wins if they can always respond.

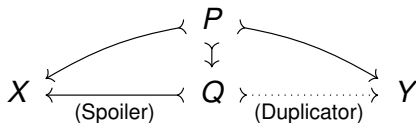
Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$(X, Y \in \mathcal{A})$

- Positions are spans of “embeddings” $(P \text{ path})$
- Moves: (played by Spoiler and Duplicator)



or symmetrically.

- Duplicator wins if they can always respond.

Definition

$X, Y \in \mathcal{A}$ are **back-and-forth equivalent** if Duplicator wins $\mathcal{G}(X, Y)$.

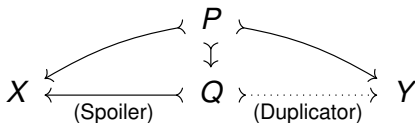
Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$(X, Y \in \mathcal{A})$

- Positions are spans of “embeddings” (P path)
- Moves: (played by Spoiler and Duplicator)



or symmetrically.

- Duplicator wins if they can always respond.

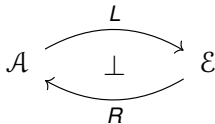
Definition

$X, Y \in \mathcal{A}$ are **back-and-forth equivalent** if Duplicator wins $\mathcal{G}(X, Y)$.

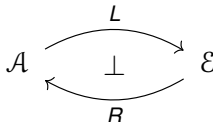
- Bisimulation via open maps. (Joyal, Nielsen, Winskel)

Abramsky & Reggio (2021, 2023).

Our goal



Our goal



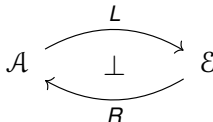
Example (Ehrenfeucht-Fraïssé games)

Arboreal $\mathcal{A}$ with right adjoint $R_{\text{EF}} : \mathbf{Struct}(\sigma) \rightarrow \mathcal{A}$ such that

M, N are $\mathcal{L}_\infty(\sigma)$ -equivalent $\iff$

$R_{\text{EF}}(M), R_{\text{EF}}(N)$ are back-and-forth equivalent

Our goal



Example (Ehrenfeucht-Fraïssé games)

Arboreal $\mathcal{A}$ with right adjoint $R_{\text{EF}} : \mathbf{Struct}(\sigma) \rightarrow \mathcal{A}$ such that

M, N are $\mathcal{L}_\infty(\sigma)$ -equivalent $\iff$

$R_{\text{EF}}(M), R_{\text{EF}}(N)$ are back-and-forth equivalent

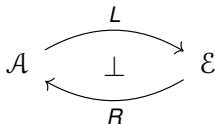
Goal

Give sufficient conditions on $L : \mathcal{A} \rightleftarrows \mathcal{E} : R$ so that

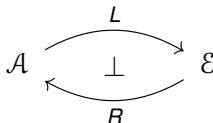
$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty$ -equivalent $\implies$

$R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

A “structure theorem” for arboreal adjunctions



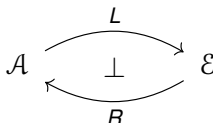
A “structure theorem” for arboreal adjunctions



In many examples:

- ▶ $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
- ▶ the right $R: \mathcal{E} \rightarrow \mathcal{A}$ adjoint is finitary,

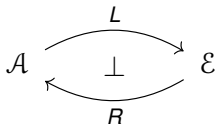
A “structure theorem” for arboreal adjunctions



In many examples:

- ▶ $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
- ▶ the right $R: \mathcal{E} \rightarrow \mathcal{A}$ adjoint is finitary,
- ▶ the paths P of $\mathcal{A}$ are finitely presentable,

A “structure theorem” for arboreal adjunctions

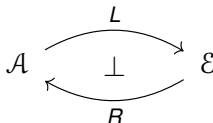


In many examples:

- ▶ $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
- ▶ the right $R: \mathcal{E} \rightarrow \mathcal{A}$ adjoint is finitary,
- ▶ the paths P of $\mathcal{A}$ are finitely presentable,
- ▶ given $f: P \rightarrow X$ in $\mathcal{A}$,

f “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$

A “structure theorem” for arboreal adjunctions



In many examples:

- ▶ $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
- ▶ the right $R: \mathcal{E} \rightarrow \mathcal{A}$ adjoint is finitary,
- ▶ the paths P of $\mathcal{A}$ are finitely presentable,
- ▶ given $f: P \rightarrow X$ in $\mathcal{A}$,

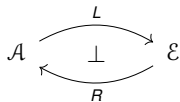
f “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$

Theorem (Reggio & R)

$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$ -equivalent $\implies$

$R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

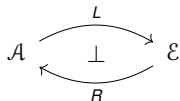
Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.

Proof

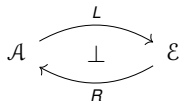


- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.

▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)

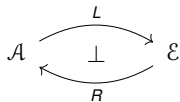
Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)

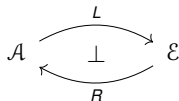
Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)
- ▶ Left adjoint $L: \mathcal{A} \rightarrow \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A})$.
(Hodges’ word-constructions (1974, 1975))

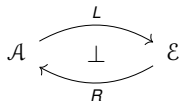
Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)
- ▶ Left adjoint $L: \mathcal{A} \rightarrow \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A})$.
(Hodges’ word-constructions (1974, 1975))
- ▶ Hintikka formulae in $\mathcal{L}_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
(define ordinal ranks of positions in games)

Proof



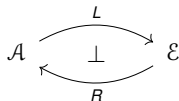
- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)
- ▶ Left adjoint $L: \mathcal{A} \rightarrow \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A})$.
(Hodges’ word-constructions (1974, 1975))
- ▶ Hintikka formulae in $\mathcal{L}_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
(define ordinal ranks of positions in games)

Lemma

If X, Y are equivalent in $\mathcal{L}_\infty(\mathcal{A})$, then X, Y are back-and-forth equivalent in $\mathcal{A}$.

Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

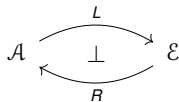
- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)
- ▶ Left adjoint $L: \mathcal{A} \rightarrow \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A})$.
(Hodges’ word-constructions (1974, 1975))
- ▶ Hintikka formulae in $\mathcal{L}_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
(define ordinal ranks of positions in games)

Lemma

If X, Y are equivalent in $\mathcal{L}_\infty(\mathcal{A})$, then X, Y are back-and-forth equivalent in $\mathcal{A}$.

- ▶ $R: \mathcal{E} \rightarrow \mathcal{A}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{A}) \rightarrow \mathcal{L}_\infty(\mathcal{E})$.

Proof



- ▶ $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- ▶ finitary right-adjoint $R: \mathcal{E} \rightarrow \mathcal{A}$,
- ▶ paths P of $\mathcal{A}$ finitely presentable.

- ▶ $f: P \rightarrow X$ “embedding” in $\mathcal{A}$ $\iff L(f)$ embedding of structures in $\mathcal{E}$.
- ▶ $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- ▶ Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
(functorial semantics and Yoneda lemma)
- ▶ Left adjoint $L: \mathcal{A} \rightarrow \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A})$.
(Hodges’ word-constructions (1974, 1975))
- ▶ Hintikka formulae in $\mathcal{L}_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
(define ordinal ranks of positions in games)

Lemma

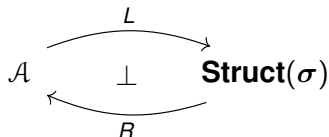
If X, Y are equivalent in $\mathcal{L}_\infty(\mathcal{A})$, then X, Y are back-and-forth equivalent in $\mathcal{A}$.

- ▶ $R: \mathcal{E} \rightarrow \mathcal{A}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{A}) \rightarrow \mathcal{L}_\infty(\mathcal{E})$.

Theorem

If M, N are equivalent in $\mathcal{L}_\infty(\mathcal{E})$, then $R(M), R(N)$ are back-and-forth equivalent in $\mathcal{A}$.

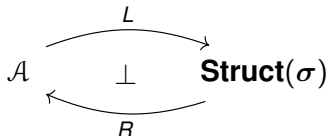
An application



Theorem

$M, N \in \mathbf{Struct}(\sigma)$ are $\mathcal{L}_\infty(\sigma)$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

An application



Theorem

$M, N \in \mathbf{Struct}(\sigma)$ are $\mathcal{L}_\infty(\sigma)$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

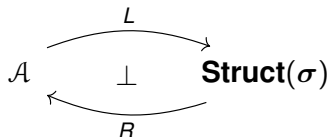
Example.

- ▶ $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\mathcal{L}_\infty(<)$ -equivalent.
- ▶ $R(\mathbb{Q}), R(\mathbb{R})$ are back-and-forth equivalent in $\mathcal{A}$.

Remark.

- ▶ Many non-isomorphic $\mathcal{L}_\infty(<)$ -equivalent structures.

An application



Theorem

$M, N \in \mathbf{Struct}(\sigma)$ are $\mathcal{L}_\infty(\sigma)$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

Example.

- ▶ $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\mathcal{L}_\infty(<)$ -equivalent.
- ▶ $R(\mathbb{Q}), R(\mathbb{R})$ are back-and-forth equivalent in $\mathcal{A}$.

Remark.

- ▶ Many non-isomorphic $\mathcal{L}_\infty(<)$ -equivalent structures.

Game comonad for MSO.

(Jackl, Marsden & Shah, 2022)

- ▶ $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not **MSO**($<$)-equivalent.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- ▶ General conditions on $R: \mathcal{E} \rightarrow \mathcal{A}$ for
 $M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- ▶ General conditions on $R: \mathcal{E} \rightarrow \mathcal{A}$ for
 $M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent
- ▶ Restricts to finite games and finitary logic.
- ▶ Covers various examples.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- ▶ General conditions on $R: \mathcal{E} \rightarrow \mathcal{A}$ for
 $M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent
- ▶ Restricts to finite games and finitary logic.
- ▶ Covers various examples.

Future work.

- ▶ Higher presentability ranks.
 (Lindström quantifiers (via the games of (Caicedo 1980)))
 (Comonadic modal logic)
- ▶ Convey stronger invariants?
 (E.g. finite variable constraint for pebble games)

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- ▶ General conditions on $R: \mathcal{E} \rightarrow \mathcal{A}$ for
 $M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$ -equivalent $\implies$
 $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent
- ▶ Restricts to finite games and finitary logic.
- ▶ Covers various examples.

Future work.

- ▶ Higher presentability ranks.
 (Lindström quantifiers (via the games of (Caicedo 1980)))
 (Comonadic modal logic)
- ▶ Convey stronger invariants?
 (E.g. finite variable constraint for pebble games)

Thanks for your attention!