Finitely accessible arboreal adjunctions and Hintikka formulae

Luca Reggio & Colin Riba

Department of Mathematics, Università degli Studi di Milano, Italy
LIP, ENS de Lyon, France

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A reformulation of an old example (1/2)

Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (1/2)

Situation

$$\langle \overline{x} \mid \varphi \rangle$$

where

- $\overline{x} = x_1, \ldots, x_n$
- φ finite conjunction of contraints $((x_i < x_j)$ or $(x_i = x_j))$

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Situation

$$\langle \overline{x} \mid \varphi \rangle \xrightarrow{m} (M, <_M)$$

where

- $\overline{x} = x_1, \ldots, x_n$
- φ finite conjunction of contraints $((x_i < x_j) \text{ or } (x_i = x_j))$
- m is an order embedding:

$$\begin{aligned} m(x_i) <_M m(x_j) \quad &\iff \quad (x_i < x_j) \text{ in } \varphi \\ m(x_i) = m(x_j) \quad &\iff \quad (x_i = x_j) \text{ in } \varphi \end{aligned}$$

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A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

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Ehrenfeucht-Fraïssé game (played by Spoiler and Duplicator)

$$\begin{tikzcd} & \langle \overline{x} \mid \varphi \rangle \arrow[dl, tail, bend right] \arrow[dr, tail, bend left] & \\ M & & N \end{tikzcd}$$

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\begin{tikzcd}
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M & \langle \overline{x}, x' \mid \varphi \wedge \varphi' \rangle \arrow[l, tail, "\text{(Spoiler)}"'] & N
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or symmetrically.

- Duplicator wins since they can always respond.

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Corollary (…, Karp (1965))

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_\infty(<)$.

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Toward game comonads

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Toward game comonads: turn plays into structures

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Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play

$$\begin{tikzcd}
 & \langle\rangle \arrow[d, tail] \arrow[ddddl, bend right, tail] \arrow[ddddr, bend left, tail] & \\
 & \langle x_1 \mid \varphi_1\rangle \arrow[d, tail] \arrow[dddl, bend right, tail] \arrow[dddr, bend left, tail] & \\
 & \langle x_1, x_2 \mid \varphi_1 \wedge \varphi_2\rangle \arrow[d, dotted, tail] \arrow[ddl, bend right, tail] \arrow[ddr, bend left, tail] & \\
 & \langle x_1, \dots, x_n \mid \varphi_1 \wedge \cdots \wedge \varphi_n\rangle \arrow[dl, tail] \arrow[dr, tail] & \\
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Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on M

$$
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 & \\
 & \langle x_1, \ldots, x_n \mid \varphi_1 \wedge \cdots \wedge \varphi_n \rangle \arrow[dl, tail] \\
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$$

is an element of a structure $R_{\mathbb{EF}}(M)$ with carrier M^*.

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Other examples

- Pebble games.
- Modal fragment, Hybrid fragment, Guarded fragments, ...

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Adjunctions

- The $R(M)$ are structures with a tree order.

$$\begin{tikzcd} \mathcal{A} & & \mathbf{Struct}(\sigma) \arrow[ll, bend left, "R"] \end{tikzcd}$$

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Other examples

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Adjunctions

- The $R(M)$ are structures with a tree order.
- In each case, R is a right adjoint.
- Comonads on $\mathbf{Struct}(\sigma)$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Struct}(\sigma) \arrow[ll, bend left, "R"] \end{tikzcd}$$

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Arboreal categories

Abramsky & Reggio (2021, 2023).

Arboreal categories: motivations

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Conditions on $\mathcal{A}$ which yield well-behaved games.

Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

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Arboreal category $\mathcal{A}$.

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
  each morphism f factors as ($e \in \mathcal{Q}$, $m \in \mathcal{M}$)

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & \bullet \\ & \bullet \arrow[ur, tail, "m"'] & \end{tikzcd}$$

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- $P \in \mathcal{A}$ is a path when its $\mathcal{M}$-subobjects form a finite chain

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- Induced functor $\mathcal{A} \to$ **Tree**.

Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$. $(X, Y \in \mathcal{A})$

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Definition

$X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $\mathcal{G}(X, Y)$.

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Definition

$X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $\mathcal{G}(X, Y)$.

- Bisimulation via open maps. (Joyal, Nielsen, Winskel)

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Our goal

$$
\begin{tikzcd}
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Example (Ehrenfeucht-Fraïssé games)

Arboreal $\mathcal{A}$ with right adjoint $R_{\mathbb{EF}} : \mathbf{Struct}(\boldsymbol{\sigma}) \to \mathcal{A}$ such that

M, N are $\mathcal{L}_\infty(\boldsymbol{\sigma})$-equivalent $\iff$

$R_{\mathbb{EF}}(M), R_{\mathbb{EF}}(N)$ are back-and-forth equivalent

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Goal

Give sufficient conditions on $L\colon \mathcal{A} \rightleftarrows \mathcal{E} : R$ so that

$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty$-equivalent $\implies$

$R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

A "structure theorem" for arboreal adjunctions

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathcal{E} \arrow[ll, bend left, "R"] \end{tikzcd}$$

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- $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
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- given $f\colon P \to X$ in $\mathcal{A}$,

$$f \text{ “embedding” in } \mathcal{A} \quad \Longleftrightarrow \quad L(f) \text{ embedding of structures in } \mathcal{E}$$

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Theorem (Reggio & R)

$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$-equivalent $\Longrightarrow$ $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

Proof

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "R"] \end{tikzcd}$$

- $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- finitary right-adjoint $R\colon \mathcal{E} \to \mathcal{A}$,
- paths P of $\mathcal{A}$ finitely presentable.

- $f\colon P \to X$ "embedding" in $\mathcal{A}$ $\iff$ $L(f)$ embedding of structures in $\mathcal{E}$.

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- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)

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- Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.

  (functorial semantics and Yoneda lemma)

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- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
  (functorial semantics and Yoneda lemma)
- Left adjoint $L\colon \mathcal{A} \to \mathcal{E}$ induces a formula translation $\mathcal{L}_\infty(\mathcal{E}) \to \mathcal{L}_\infty(\mathcal{A})$.
  (Hodges' word-constructions (1974, 1975))

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Theorem

If M, N are equivalent in $\mathcal{L}_\infty(\mathcal{E})$, then $R(M), R(N)$ are back-and-forth equivalent in $\mathcal{A}$.

An application

$$\begin{tikzcd} \mathcal{A} \arrow[rr, phantom, "\perp"] \arrow[rr, leftarrow, bend right, "R"'] & & \mathbf{Struct}(\sigma) \arrow[ll, leftarrow, bend right, "L"'] \end{tikzcd}$$

Theorem

$M, N \in \mathbf{Struct}(\sigma)$ are $\mathcal{L}_\infty(\sigma)$-equivalent $\implies$ *$R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent*

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Example.

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\mathcal{L}_\infty(<)$-equivalent.
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- Many non-isomorphic $\mathcal{L}_\infty(<)$-equivalent structures.

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Game comonad for MSO. (Jackl, Marsden & Shah, 2022)

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not **MSO**$(<)$-equivalent.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- General conditions on $R\colon \mathcal{E} \to \mathcal{A}$ for

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Thanks for your attention!