Lax comma categories: descent and exponentiability

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Lax comma categories

Let $\mathbb{A}$ be a 2-category, X an object of $\mathbb{A}$, $\mathbb{B}$ a full sub-2-category of $\mathbb{A}$.

The lax comma (2-)category $\mathbb{B} \Downarrow X$ has

- objects: morphisms $f\colon B \to X$ of $\mathbb{A}$ with B in $\mathbb{B}$,
- morphisms $f \to g$: 2-cells θ of $\mathbb{A}$ of the form

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

where b is a morphism in $\mathbb{B}$.

Examples

Let $\mathbb{A} = \mathsf{CAT}$, $\mathbb{B} = \mathsf{Set}$, and $\mathcal{X}$ a category.

The category $\mathsf{Set} \Downarrow \mathcal{X}$ consists of

- objects: functors $\mathcal{A} \to \mathcal{X}$,
- morphisms $F \to G$: natural transformations

$$\begin{tikzcd} A \arrow[rr, "H"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[dl, "G"] \\ & X & \end{tikzcd}$$

We note $\mathsf{Cat} \Downarrow \mathcal{X}$ is the category of diagrams on $\mathcal{X}$.

Examples

Let $\mathbb{A} = \mathbb{B} = \mathsf{Ord}$, and X an ordered set.

The category $\mathsf{Ord} \Downarrow X$ consists of

- objects: monotone maps $\alpha \colon A \to X$, that is, ordered families $(\alpha(a))_{a \in A}$ of elements $\alpha(a) \in X$.
- morphisms $\alpha \to \beta$: a monotone map $f \colon A \to B$ satisfying

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

  that is, $\alpha(a) \leqslant \beta(f(a))$ for all $a \in A$.

Examples

Let $\mathbb{A} = \mathsf{CAT}$, $\mathbb{B} = \mathsf{Cat}$, and $\mathcal{X}$ a (possibly large) category.

The category $\mathsf{Cat} \Downarrow X$ consists of

- objects: functors $: A \to X$, that is, ordered families $(\alpha(a))_{a \in A}$ of elements $\alpha(a) \in X$.
- morphisms $\alpha \to \beta$: a monotone map $f\colon A \to B$ satisfying

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

  that is, $\alpha(a) \leqslant \beta(f(a))$ for all $a \in A$.

Grothendieck construction

The lax comma category $\mathbb{A} \Downarrow X$ is the total category of *Grothendieck construction* of the 2-functor

$$\begin{aligned} \mathbb{A}^{\mathsf{op}} &\to \mathsf{CAT} \\ A &\mapsto \mathbb{B}(A, X) \\ f\colon A \to B &\mapsto - \cdot f\colon \mathbb{B}(B, X) \to \mathbb{B}(A, X) \end{aligned}$$

that is, we have a fibration $\mathbb{A} \Downarrow X \to \mathbb{A}$.

Consequences

Properties of X determine the properties of $\mathbb{A} \Downarrow X$:

- Existence of limits/completeness (Gray 1966)
- Existence of colimits/cocompleteness (Gray 1966)
- Distributivity of colimits over limits (Clementino, Lucatelli Nunes, P. 2024)
- Topologicity (Wyler 1981, Clementino, Lucatelli Nunes, P. 2024)
- Effective descent morphisms (Clementino, Lucatelli Nunes, P. 2024)
- Exponentiable objects (Clementino, Lucatelli Nunes, P. 2024)

Cartesian closedness

Theorem (Lucatelli Nunes, Vákár 2024)

Let $\mathcal{X}$ be a infinitary distributive category. The following are equivalent:

- $\mathcal{X}$ is cartesian closed.
- $\mathsf{Fam}(\mathcal{X})$ is cartesian closed.

Cartesian closedness

Theorem (Clementino, Lucatelli Nunes 2023)

We consider the lax comma category $\mathsf{Ord} \Downarrow X$, for X a complete ordered set.
The following are equivalent:

- X is cartesian closed.
- $\mathsf{Ord} \Downarrow X$ is cartesian closed.

Theorem (Clementino, Lucatelli Nunes, P. 2024)

We consider the lax comma category $\mathsf{Cat} \Downarrow \mathcal{X}$, for $\mathcal{X}$ a category.

If $\mathcal{X}$ is cartesian closed, then so is $\mathsf{Cat} \Downarrow \mathcal{X}$.

Effective descent morphisms

Let $\mathcal{C}$ be a category with pullbacks.

$$p^* \colon \mathcal{C} \downarrow y \to \mathcal{C} \downarrow x$$

We say p is an *effective descent morphism* (descent morphism) if p^* is monadic (premonadic).

Effective descent morphisms

We consider a lax comma category $\mathbb{A} \Downarrow X$.

Theorem (Lucatelli Nunes, P. 2024)

If X satisfies mild conditions, then

$$\mathbb{A} \Downarrow X \to \mathbb{A}$$

preserves effective descent morphisms.

Effective descent morphisms

Let X be an ordered set whose downsets $\downarrow x$ are complete lattices.

Theorem (P, 2024)

The descent morphisms in $\mathsf{Fam}(X)$ are effective for descent.

We consider the lax comma category $\mathsf{Ord} \Downarrow X$.

Theorem (Clementino, P. 2024)

A morphism $f\colon (\alpha(a))_{a \in A} \to (\beta(b))_{b \in B}$ is an effective descent morphism if and only if

- $f\colon A \to B$ is an effective descent morphism in Ord,
- $f\colon (\alpha(a))_{a \leqslant a'} \to (\beta(b))_{b \leqslant b'}$ is a descent morphism in $\mathsf{Fam}(X)$.

On-going work

- Studying exponentiable objects and effective descent morphisms in $\mathsf{Top} \Downarrow X$.
- Characterization of effective descent morphisms in $\mathsf{Cat} \Downarrow X$.

Dank wel!