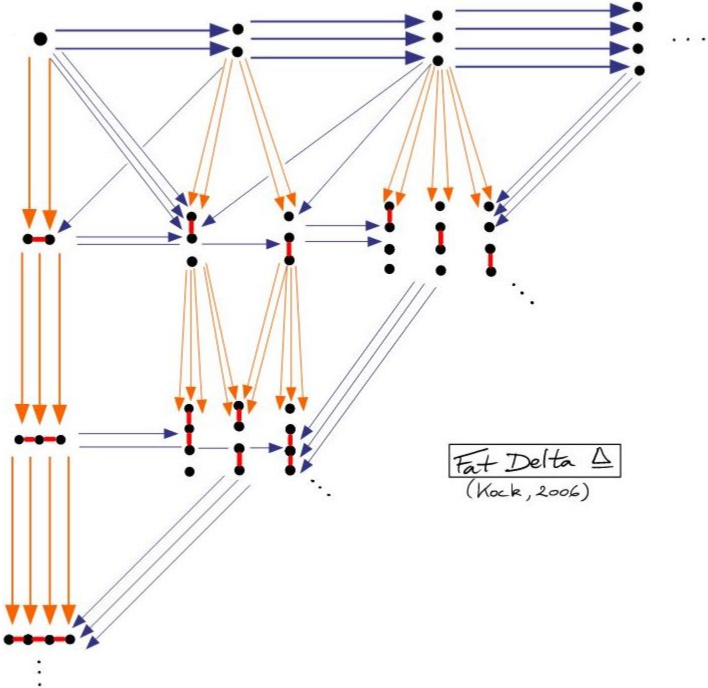
A study of Kock's fat Delta

Tom de Jong[1] Nicolai Kraus[1] Simona Paoli[2] Stiéphen Pradal[1]

[1]University of Nottingham [2]University of Aberdeen

17th November 2024, 109th PSSL, Leiden



Fat Delta Δ
(Kock, 2006)

Motivation: Simpson's conjecture

- Motivation for Kock (2006) to introduce $\underline{\Delta}$:
  - The simplex category Δ with **degeneracies up to homotopy**
  - The identity coherence structure is part of the data as objects
- Motivation, in low dimension, for Paoli (2024) to study $\underline{\Delta}$ further

Introduction of $\underline{\Delta}$

Motivation: Simpson's conjecture

- Motivation for Kock (2006) to introduce $\underline{\Delta}$:
  - The simplex category Δ with **degeneracies up to homotopy**
  - The identity coherence structure is part of the data as objects
- Motivation, in low dimension, for Paoli (2024) to study $\underline{\Delta}$ further

Motivation: higher categories in homotopical type theories

- Construct Reedy fibrant diagrams over direct categories
- Use simplicial methods

Obstacle: Δ is not a direct category $\rightsquigarrow$ Need of a direct replacement

- Introduction of a variation of $\underline{\Delta}$ by Kraus and Sattler (2017)

There are other issues: working with degeneracies might be problematic in a constructive framework, as explained by Sattler (2018).

Developing properties of $\underline{\underline{\Delta}}$

We have developed

- Factorisation systems
- Nerve theorem
- Generators and relations

Developing properties of $\underline{\Delta}$

We have developed

- **Factorisation systems**
- **Nerve theorem**
- Generators and relations

In this talk we will focus on how we get the two first points.

Definition of $\underline{\Delta}$

Definition (Relative semicategories and semiordinals)

Semicategory	=	Category w/o identity structure
Relative semicategory	=	Semicategory w/ wide subsemicategory
Semiordinal	=	Total strict order (viewed as a semicategory)

Definition of $\underline{\Delta}$

Definition (Relative semicategories and semiordinals)

Semicategory	=	Category w/o identity structure
Relative semicategory	=	Semicategory w/ wide subsemicategory
Semiordinal	=	Total strict order (viewed as a semicategory)

Definition (Fat Delta $\underline{\Delta}$, Kock 2006)

The category $\underline{\Delta}$ is the category of **relative finite non-empty semiordinals** and maps between them.

Definition of $\underline{\Delta}$

Definition (Relative semicategories and semiordinals)

Semicategory	=	Category w/o identity structure
Relative semicategory	=	Semicategory w/ wide subsemicategory
Semiordinal	=	Total strict order (viewed as a semicategory)

Definition (Fat Delta $\underline{\Delta}$, Kock 2006)

The category $\underline{\Delta}$ is the category of **relative finite non-empty semiordinals** and maps between them.

Examples

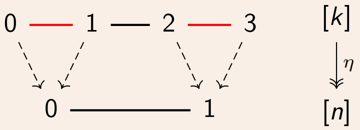
- $0 \rightarrow 1$
- $0 \rightarrow 1 \rightarrow 2$
- $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$

The category $\underline{\Delta}$ also admits another description: $\mathrm{Epi}(\Delta)_{\mathrm{Mono}}$

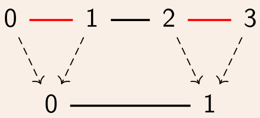
Objects	Morphisms

The category $\underline{\Delta}$ also admits another description: $\mathrm{Epi}(\Delta)_{\mathrm{Mono}}$

Objects	Morphisms
Epimorphisms in Δ	

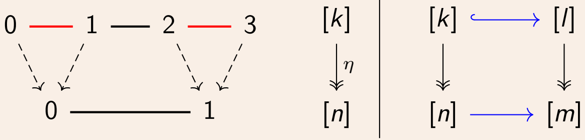


The category $\underline{\Delta}$ also admits another description: $\mathrm{Epi}(\Delta)_{\mathrm{Mono}}$

Objects	Morphisms
Epimorphisms in Δ	Squares of the form
 $\eta : [k] \twoheadrightarrow [n]$	$\begin{array}{ccc} [k] & \hookrightarrow & [l] \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ [n] & \longrightarrow & [m] \end{array}$

The category $\underline{\Delta}$ also admits another description: $\mathrm{Epi}(\Delta)_{\mathrm{Mono}}$

Objects	Morphisms
Epimorphisms in Δ	Squares of the form



$\rightsquigarrow$ Investigation using the theory of **monads with arities**.

Monad with arities: Idea

- Generalized nerve constructions
- Algebras as presheaves with (Segal) conditions
- Correspondence: monads $\longleftrightarrow$ algebraic theories

Monad with arities: Idea

- Generalized nerve constructions
- Algebras as presheaves with (Segal) conditions
- Correspondence: monads $\longleftrightarrow$ algebraic theories

Example: The simplex category Δ

Let $\mathsf{fc} : \mathsf{Graph} \to \mathsf{Graph}$ be the free category monad on directed graphs, then

- arities $= \Delta_0$ (the wide subcategory of distance-preserving maps)
- the associated theory $\Theta_{\mathsf{fc}} = \Delta$
- the nerve $\mathcal{N} : \mathsf{Cat} \to \widehat{\Delta}$ is fully faithful
- small categories $=$ simplicial sets satisfying the Segal condition
- active-inert factorisation system.

Monad with arities: Definition

Let $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$ with $\mathcal{A}$ dense, recall that

- every object X can be identified with the canonical colimit over the slice $\mathcal{A}_{/X}$
- the $\mathcal{A}$-nerve functor $\mathcal{N}_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}},\ X \mapsto \mathcal{E}(i_{\mathcal{A}}, X)$ is fully faithful.

Monad with arities: Definition

Let $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$ with $\mathcal{A}$ dense, recall that

- every object X can be identified with the canonical colimit over the slice $\mathcal{A}_{/X}$
- the $\mathcal{A}$-nerve functor $\mathcal{N}_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}},\ X \mapsto \mathcal{E}(i_{\mathcal{A}}, X)$ is fully faithful.

Definition (Monad with arities, Weber 2007)

A monad (T, μ, ν) on $\mathcal{E}$ has arities $\mathcal{A}$ if the functor $\mathcal{N}_{\mathcal{A}}\, T$ **preserves** the canonical colimits over slices $\mathcal{A}_{/X}$.

Monad with arities: Definition

Let $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$ with $\mathcal{A}$ dense, recall that

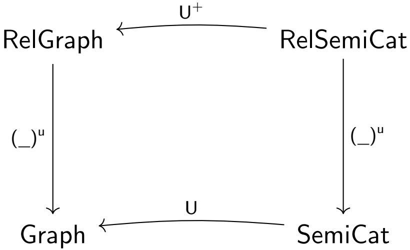
- every object X can be identified with the canonical colimit over the slice $\mathcal{A}_{/X}$
- the $\mathcal{A}$-nerve functor $\mathcal{N}_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}},\ X \mapsto \mathcal{E}(i_{\mathcal{A}}, X)$ is fully faithful.

Definition (Monad with arities, Weber 2007)

A monad (T, μ, ν) on $\mathcal{E}$ has arities $\mathcal{A}$ if the functor $\mathcal{N}_{\mathcal{A}}\, T$ **preserves** the canonical colimits over slices $\mathcal{A}_{/X}$.

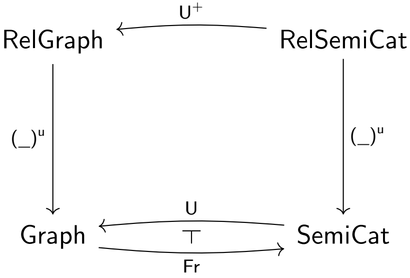
$\rightsquigarrow$ We rather work with a sufficient property: **strongly cartesian** monads.

Using forgetful functors we can construct the following diagram:



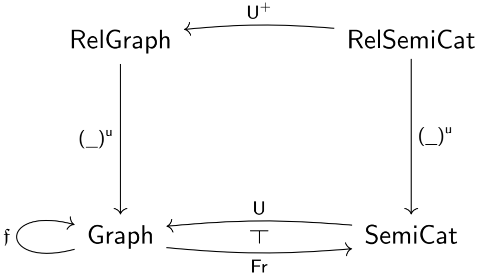
Step 1: Construction of $\mathfrak{f}^+$

Using forgetful functors we can construct the following diagram:



Step 1: Construction of $\mathfrak{f}^+$

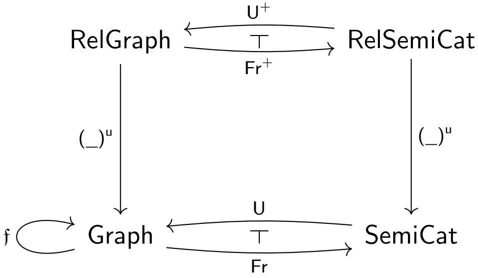
Using forgetful functors we can construct the following diagram:



- $\mathfrak{f}\, G = ([n] \to G, +)$

Step 1: Construction of $\mathfrak{f}^+$

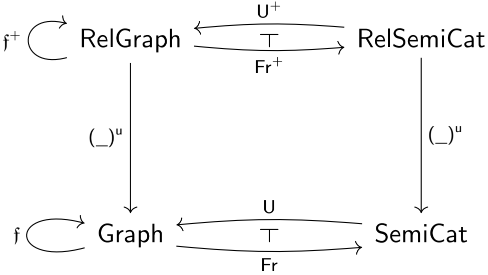
Using forgetful functors we can construct the following diagram:



- $\mathfrak{f}\, G = ([n] \to G, +)$

Step 1: Construction of $\mathfrak{f}^+$

Using forgetful functors we can construct the following diagram:



- $\mathfrak{f}\, G = ([n] \to G, +)$
- $\mathfrak{f}^+(G^+ \hookrightarrow G) = \mathfrak{f}\, G^+ \hookrightarrow \mathfrak{f}\, G$

Step 2: $\mathfrak{f}$ and $\mathfrak{f}^+$ are strongly cartesian

Definition (Strongly cartesian monad, Street 2000)

A monad (T, μ, ν) on category $\mathcal{E}$ with a terminal object 1 is strongly cartesian if it is

cartesian: $\mathcal{E}$ has pullbacks, T **preserves** them, and μ and ν are **cartesian**

local right adjoint: the functor $T_1 : \mathcal{E} \to \mathcal{E}_{/T1}$ has a **left adjoint**.

Step 2: $\mathfrak{f}$ and $\mathfrak{f}^{+}$ are strongly cartesian

Definition (Strongly cartesian monad, Street 2000)

A monad (T, μ, ν) on category $\mathcal{E}$ with a terminal object 1 is strongly cartesian if it is

cartesian: $\mathcal{E}$ has pullbacks, T **preserves** them, and μ and ν are **cartesian**

local right adjoint: the functor $T_1 : \mathcal{E} \to \mathcal{E}_{/T1}$ has a **left adjoint**.

The second condition is equivalent to asking that each $f : A \to T1$ **factors** as

$$A \xrightarrow{g} TX \xrightarrow{T!_1} T1$$

where g has the following property (T-generic):

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & TX' \\ {\scriptstyle g}\downarrow & \overset{T\delta}{\nearrow} & \downarrow{\scriptstyle T\gamma} \\ TX & \xrightarrow[T\beta]{} & TY \end{array}$$

such that $\gamma\delta = \beta$ and $T\delta g = \alpha$.

Proposition

The monad $\mathfrak{f} : \mathsf{Graph} \to \mathsf{Graph}$ *is strongly cartesian.*

Proposition

The monad $\mathfrak{f} : \mathrm{Graph} \to \mathrm{Graph}$ *is strongly cartesian.*

Sketch of the proof.

It boils down to proving that the morphism $\hat{n} : [1] \to \mathfrak{f}[n]$ mapping $[1]$ to the **maximal path** $\mathrm{id}_{[n]}$ is $\mathfrak{f}$-generic:

$$\begin{array}{ccc} [1] & \xrightarrow{\alpha} & \mathfrak{f}\,X \\ {\scriptstyle \hat{n}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}\,\gamma} \\ \mathfrak{f}[n] & \xrightarrow[\mathfrak{f}\,\beta]{} & \mathfrak{f}\,Y \end{array}$$

□

Proposition

The monad $\mathfrak{f} : \mathsf{Graph} \to \mathsf{Graph}$ *is strongly cartesian.*

Sketch of the proof.

It boils down to proving that the morphism $\hat{n} : [1] \to \mathfrak{f}[n]$ mapping $[1]$ to the **maximal path** $\mathrm{id}_{[n]}$ is $\mathfrak{f}$-generic:

$$\begin{array}{ccc} [1] & \xrightarrow{\alpha} & \mathfrak{f}\,X \\ {\scriptstyle \hat{n}}\downarrow & \overset{\mathfrak{f}\,\delta}{\nearrow} & \downarrow{\scriptstyle \mathfrak{f}\,\gamma} \\ \mathfrak{f}[n] & \xrightarrow[\mathfrak{f}\,\beta]{} & \mathfrak{f}\,Y \end{array}$$

Define $\delta : [n] \to X$ as the image of α. □

Proposition

The monad $\mathfrak{f} : \mathsf{Graph} \to \mathsf{Graph}$ *is strongly cartesian.*

Sketch of the proof.

It boils down to proving that the morphism $\hat{n} : [1] \to \mathfrak{f}[n]$ mapping $[1]$ to the **maximal path** $\mathrm{id}_{[n]}$ is $\mathfrak{f}$-generic:

$$
\begin{array}{ccc}
[1] & \xrightarrow{\alpha} & \mathfrak{f}\,X \\
{\scriptstyle \hat{n}}\downarrow & \overset{\mathfrak{f}\,\delta}{\nearrow} & \downarrow{\scriptstyle \mathfrak{f}\,\gamma} \\
\mathfrak{f}[n] & \xrightarrow[\mathfrak{f}\,\beta]{} & \mathfrak{f}\,Y
\end{array}
$$

Define $\delta : [n] \to X$ as the image of α. □

Proposition

The monad $\mathfrak{f}^{+} : \mathsf{RelGraph} \to \mathsf{RelGraph}$ *is strongly cartesian.*

Step 3: Construction of the arities

Theorem (Berger, Melliès, and Weber 2012)

Suppose T is a strongly cartesian monad on $\mathcal{E}$ (finitely complete) with a dense generator $\mathcal{A}$. Then T is a monad with arities $\mathcal{A}_T$ given by taking the full subcategory of $\mathcal{E}$ **generated by the T-generic factorisations** $A \to TA_T \to T1$.

Step 3: Construction of the arities

Theorem (Berger, Melliès, and Weber 2012)

Suppose T is a strongly cartesian monad on $\mathcal{E}$ (finitely complete) with a dense generator $\mathcal{A}$. Then T is a monad with arities $\mathcal{A}_T$ given by taking the full subcategory of $\mathcal{E}$ **generated by the T-generic factorisations** *$A \to TA_T \to T1$.*

Definition (The dense subcategory $\mathbb{A}$)

The category $\mathbb{A}$ consists of **alternatingly marked linear graphs** $\mathfrak{n}^0 = [1]^\sharp + [1]^\flat + \cdots + [1]^{\sigma_n^0}$ and $\mathfrak{n}^1 = [1]^\flat + [1]^\sharp + \cdots + [1]^{\sigma_n^1}$ and relative graph morphisms between them.

Step 3: Construction of the arities

Theorem (Berger, Melliès, and Weber 2012)

Suppose T is a strongly cartesian monad on $\mathcal{E}$ (finitely complete) with a dense generator $\mathcal{A}$. Then T is a monad with arities $\mathcal{A}_T$ given by taking the full subcategory of $\mathcal{E}$ **generated by the T-generic factorisations** *$A \to TA_T \to T1$.*

Definition (The dense subcategory $\mathbb{A}$)

The category $\mathbb{A}$ consists of **alternatingly marked linear graphs** $\mathfrak{n}^0 = [1]^\sharp + [1]^\flat + \cdots + [1]^{\sigma_n^0}$ and $\mathfrak{n}^1 = [1]^\flat + [1]^\sharp + \cdots + [1]^{\sigma_n^1}$ and relative graph morphisms between them.

Proposition

Any map $\mathfrak{n}^\epsilon \to \mathfrak{f}^+ 1$ has a $\mathfrak{f}^+$-generic factorisation given by

$$\mathfrak{n}^\epsilon \to \mathfrak{f}^+ [m_1]^{\sigma_1^\epsilon} + \cdots + [m_n]^{\sigma_n^\epsilon} \to \mathfrak{f}^+ 1$$

Step 4: Results

Denote by $\underline{\Delta}_0$ the subcategory of $\underline{\Delta}$ consisting of relative semiordinals and **distance preserving morphisms**.

Step 4: Results

Denote by $\underline{\Delta}_0$ the subcategory of $\underline{\Delta}$ consisting of relative semiordinals and **distance preserving morphisms**.

Theorem (Arities $\underline{\Delta}_0$ and the associated theory $\underline{\Delta}$)

- *The category of arities $\underline{\Delta}_0$ is the full subcategory of* RelGraph *generated by the objects of the form $[m_1]^{\sigma_1^\epsilon} + \cdots + [m_n]^{\sigma_n^\epsilon}$ (**marked linear graphs**).*
- *The associated theory is the category of free $\mathfrak{f}^+$-algebras over $\underline{\Delta}_0$, that is the category $\underline{\Delta}$ of **relative semiordinals**.*

Step 4: Results

Denote by $\underline{\Delta}_0$ the subcategory of $\underline{\Delta}$ consisting of relative semiordinals and **distance preserving morphisms**.

Theorem (Arities $\underline{\Delta}_0$ and the associated theory $\underline{\Delta}$)

- *The category of arities $\underline{\Delta}_0$ is the full subcategory of* RelGraph *generated by the objects of the form $[m_1]^{\sigma_1^\epsilon} + \cdots + [m_n]^{\sigma_n^\epsilon}$ (**marked linear graphs**).*
- *The associated theory is the category of free $\mathfrak{f}^+$-algebras over $\underline{\Delta}_0$, that is the category $\underline{\Delta}$ of **relative semiordinals**.*

Theorem (Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$)

*The category $\underline{\Delta}$ has an **active-inert** factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$ consisting of distance-preserving and endpoint-preserving morphisms.*

Step 4: Results

Denote by $\underline{\Delta}_0$ the subcategory of $\underline{\Delta}$ consisting of relative semiordinals and **distance preserving morphisms**.

Theorem (Arities $\underline{\Delta}_0$ and the associated theory $\underline{\Delta}$)

- *The category of arities $\underline{\Delta}_0$ is the full subcategory of* RelGraph *generated by the objects of the form $[m_1]^{\sigma_1^\epsilon} + \cdots + [m_n]^{\sigma_n^\epsilon}$ (**marked linear graphs**).*
- *The associated theory is the category of free $\mathfrak{f}^+$-algebras over $\underline{\Delta}_0$, that is the category $\underline{\Delta}$ of **relative semiordinals**.*

Theorem (Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$)

*The category $\underline{\Delta}$ has an **active-inert** factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$ consisting of distance-preserving and endpoint-preserving morphisms.*

Theorem (Nerve theorem for $\underline{\mathcal{N}}$)

*The nerve functor $\underline{\mathcal{N}}$: RelSemiCat $\to \widehat{\underline{\Delta}}$ is **fully faithful**, and the essential image is spanned by the presheaves whose restriction along $\underline{\Delta}_0 \hookrightarrow \underline{\Delta}$ belongs to the essential image of $\underline{\mathcal{N}}$: RelGraph $\to \widehat{\underline{\Delta}_0}$.*

Conclusion

Contribution

- Nerve theorem for $\underline{\Delta}$
- Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$
- Generalised Segal conditions

Conclusion

Contribution

- Nerve theorem for $\underline{\underline{\Delta}}$
- Active-inert factorisation system $(\underline{\underline{\Delta}}_a, \underline{\underline{\Delta}}_0)$
- Generalised Segal conditions

- The category $\underline{\underline{\Delta}}$ enjoys many similar properties as Δ
- In particular, $\underline{\underline{\Delta}}$ is for relative semicategories what Δ is for categories

Conclusion

Contribution

- Nerve theorem for $\underline{\Delta}$
- Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$
- Generalised Segal conditions

- The category $\underline{\Delta}$ enjoys many similar properties as Δ
- In particular, $\underline{\Delta}$ is for relative semicategories what Δ is for categories

Work in progress

- $\underline{\Delta}$ as a hypermoment category (in the sense of Berger (2022))
- $\underline{\Delta}$-spaces

Conclusion

Contribution

- Nerve theorem for $\underline{\Delta}$
- Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$
- Generalised Segal conditions

- The category $\underline{\Delta}$ enjoys many similar properties as Δ
- In particular, $\underline{\Delta}$ is for relative semicategories what Δ is for categories

Work in progress

- $\underline{\Delta}$ as a hypermoment category (in the sense of Berger (2022))
- $\underline{\Delta}$-spaces

Thank you!

Bibliography

- Berger, Clemens (2022). "Moment Categories and Operads". In: *Theory and Application of Categories* 38.9. URL: http://www.tac.mta.ca/tac/volumes/38/39/38-39.pdf.
- Berger, Clemens, Paul-André Melliès, and Mark Weber (2012). "Monads with Arities and Their Associated Theories". In: *Journal of Pure and Applied Algebra* 216.8. DOI: 10.1016/j.jpaa.2012.02.039.
- Kock, Joachim (2006). "Weak Identity Arrows in Higher Categories". In: *International Mathematics Research Papers*. DOI: 10.1155/IMRP/2006/69163.
- Kraus, Nicolai and Christian Sattler (2017). *Space-Valued Diagrams, Type-Theoretically (Extended Abstract)*. DOI: 10.48550/arXiv.1704.04543.
- Paoli, Simona (2024). *Weakly Globular Double Categories and Weak Units*. DOI: 10.48550/arXiv.2008.11180.
- Sattler, Christian (2018). *Constructive homotopy theory of marked semisimplicial sets*. DOI: 10.48550/arXiv.1809.11168.
- Street, Ross (2000). "The petit topos of globular sets". In: *Journal of Pure and Applied Algebra* 154.1-3.
- Weber, Mark (2007). "Familial 2-Functors and Parametric Right Adjoints". In: *Theory and Applications of Categories* 18.