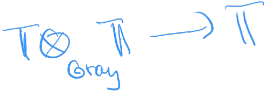

Generalized Fox's Theorem and pseudocommutativity

(work in progress, joint with John Bourke)

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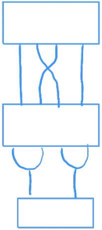

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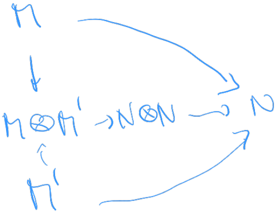


Classical Fox Theorem

Thm. (baby Fox)

Tensor product of commutative rings is their coproduct.

Thm. (adult Fox)

For any symmetric monoidal category $(\mathcal{C}, \otimes, I)$, tensor product lifts to the category $\mathsf{CMon}(\mathcal{C})$ and equips it with a cocartesian monoidal structure.

Thm. (formal Fox)

$\mathsf{CMon}(-)\colon \mathbf{SMCat} \to \mathbf{SMCat}$ is an idempotent 2-comonad and the (inclusion of the) full sub-2-category of coalgebras can be identified with $\mathbf{Cat}^{\sqcup} \to \mathbf{SMCat}$.

- cats with finite coprodts
- coproduct-preserving functors
- natural transformations

- symmetric monoidal cats
- strong symmetric monoidal functors
- monoidal natural transformations

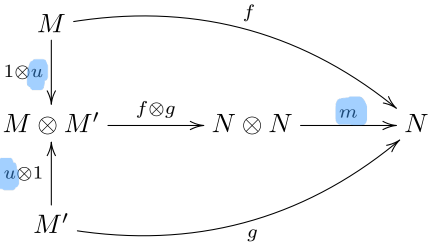
Why is this true?

Thm. (adult Fox)

For any symmetric monoidal category $(\mathcal{C}, \otimes, I)$, tensor product lifts to the category $\mathsf{CMon}(\mathcal{C})$ and equips it with a cocartesian monoidal structure.

Sketch of the proof:

1. For $M, N \in \mathsf{CMon}(\mathcal{C})$, $M \otimes N$ carries a structure of commutative monoid as well. (Need **symmetric** monoidal categories!)
2. Tensor product is a coproduct in $\mathsf{CMon}(\mathcal{C})$: need the algebraic operations to be homomorphisms!



Lax Fox to rule them all

Lemma (folklore?)

$\mathsf{CMon}(-)$ extends to an idempotent 2-comonad on $\mathbf{SMCat}^{\mathrm{lax}}$ and the coalgebras are: cats with finite coproducts, **all functors**, natural transformations.

Sketch of the proof:

1. Lax functor $F\colon \mathcal{C} \to \mathcal{D}$ lifts to $\mathsf{CMon}(\mathcal{C}) \to \mathsf{CMon}(\mathcal{D})$, in fact: $\mathsf{CMon}(\mathcal{C}) = \mathsf{Fun}^{\mathrm{lax}}(*, \mathcal{C})$.
2. Laxator $FM \otimes FN \to F(M \otimes N) \leftrightarrow$ universal map $F(M) \sqcup F(N) \to F(M \sqcup N)$.
3. We see that then F strong $\Rightarrow \mathsf{CMon}(F)$ preserves coproducts.

How to generalize?

Variants:

2-comonad T	Relaxing the Range	Rebranding T	Coalgebras
$\mathsf{CMon}(-)$	$\mathbf{SMCat}^{\mathrm{lax}}$	$\mathsf{Fun}^{lax}(*, -)$	$\mathbf{Cat}^{\sqcup}$
$\mathsf{CComon}(-)$	$\mathbf{SMCat}^{\mathrm{colax}}$	$\mathsf{Fun}^{colax}(*, -)$	$\mathbf{Cat}^{\sqcap}$
$\mathsf{CBimon}(-)$	$\mathbf{SMCat}^{\mathrm{bilax}}$	$\mathsf{Fun}^{bilax}(*, -)$	$\mathbf{Cat}^{\oplus}$

To sum it up:

1. $\mathbf{SMCat}$ = commutative pseudomonoids in $\mathbf{Cat}$.
2. We can try to replace the 2-theory for commutative pseudomonoids by any Lawvere 2-theory $\mathbb{T}$.
3. Need to define lax, colax (even bilax?) morphisms.
4. Some commutativitycondition on $\mathbb{T}$ (”operations are homomorphisms”).

Outline

Setting the stage for general $\mathbb{T}$:

1. Setting: Lawvere 2-theories, their models in **Cat**.
2. Models come with notions of strict, pseudo, lax, colax, bilax homomorphisms.
3. We obtain $\mathrm{Hom}^{\mathrm{lax}}(*, -)\colon \mathsf{Mod}(\mathbb{T}) \to \mathbf{Cat}$ and variants.

Commutativity condition:

1. Pseudocommutativity for 2-theories.
2. $\mathrm{Hom}^{\mathrm{lax}}(*, -)$ promoted to an endo-2-functor.
3. Stronger: closed multicategory structure on $\mathsf{Mod}(\mathbb{T})$ for bi/co/lax maps.

Putting it all together:

1. A formal Yoneda-type proof of 2-comonadicity.
2. Idempotence, identifying algebras using Eckmann-Hilton-based ideas.

Some proper definitions

𝔽 := skeleton of FinSet

Lawvere theory

Define Lawvere 2-theory to be an identity-on-objects 2-functor $\theta\colon \mathbb{F}^{op} \to \mathbb{T}$.

For any $w \in \{\text{strict, pseudo, lax, colax}\}$, we define the 2-category $\mathsf{Mod}_w(\mathbb{T})$ of ($\mathbf{Cat}$-valued) $\mathbb{T}$-models, w-homomorphisms as a pullback:

Finite Power-preserving functors

$$\begin{array}{ccccc}
\mathsf{Mod}_w(\mathbb{T}) & \longrightarrow & & & \mathsf{FP}(\mathbb{T}, \mathbf{Cat})_w \\
\downarrow U & & & & \downarrow \theta^* \\
\mathbf{Cat} & \xrightarrow{\simeq} & \mathsf{FP}(\mathbb{F}^{op}, \mathbf{Cat})_{\text{strict}} & \xrightarrow{\subseteq} & \mathsf{FP}(\mathbb{F}^{op}, \mathbf{Cat})_w
\end{array}$$

Pseudocommutativity

Idea: for any model X, we want all the operations $X^m \to X^n$ to be pseudo-homomorphisms of models.

Syntactic solution for 1-theories

For any $\alpha\colon m \to n$, $\beta\colon k \to l$ in $\mathbb{T}$, we have a commutative square:

$$\begin{array}{ccc} m \cdot k & \xrightarrow{m\cdot\beta} & m \cdot l \\ {\scriptstyle \alpha\cdot k}\downarrow & & \downarrow{\scriptstyle \alpha\cdot l} \\ n \cdot k & \xrightarrow[n\cdot\beta]{} & n \cdot l \end{array}$$

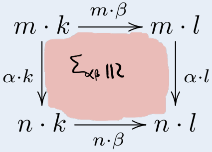
Formally: certain commutative monoid structure $\mathbb{T} \times \mathbb{T} \to \mathbb{T} \leftarrow *$.

Pseudocommutativity

Idea: for any model X, we want all the operations $X^m \to X^n$ to be pseudo-homomorphisms of models.

Syntactic solution for 2-theories

For any $\alpha\colon m \to n$, $\beta\colon k \to l$ in $\mathbb{T}$, we have a commutative square:



$\Sigma_{\alpha\beta}$ satisfying some coherence conditions

Formally: certain commutative monoid structure $\mathbb{T} \otimes \mathbb{T} \to \mathbb{T} \leftarrow *$.
Gray

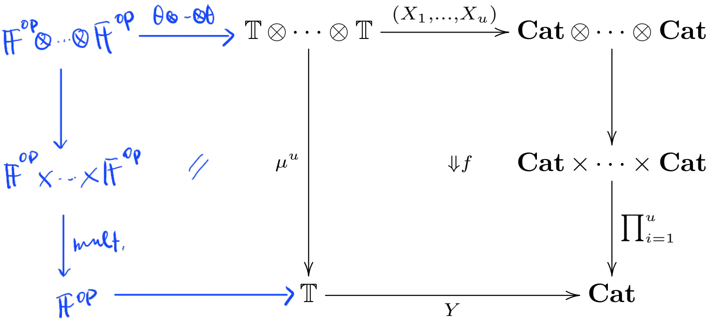
Pseudocommutativity

Let $\mathbb{T}$ be a Lawvere 2-theory. A pseudocommutativty on $\mathbb{T}$ consists of a stucture $(\mathbb{T}, \mu, u)$ of a monoid in $(\mathbf{Cat}, \otimes_{\text{Gray}}, *)$ on $\mathbb{T}$ such that

1. μ preserves products in each variable,
2. the composite $\mathbb{F}^{op} \otimes \mathbb{T} \xrightarrow{\theta\otimes} \mathbb{T} \otimes \mathbb{T} \xrightarrow{\mu} \mathbb{T}$ factors through the canonical map $\mathbb{F}^{op} \otimes \mathbb{T} \to \mathbb{F}^{op} \times \mathbb{T}$ (and the same holds for the symmetrical map $\mathbb{T} \otimes \mathbb{F}^{op} \to \mathbb{T}$),
3. the following commutes (where "mult." is the usual multiplication of natural numbers) and $u(*) = 1$.

$$\begin{array}{ccc}
\mathbb{F}^{op} \otimes \mathbb{F}^{op} & \xrightarrow{\theta\otimes\theta} & \mathbb{T} \otimes \mathbb{T} \\
\downarrow & & \\
\mathbb{F}^{op} \times \mathbb{F}^{op} & & \downarrow \mu \\
\downarrow \text{mult.} & & \\
\mathbb{F}^{op} & \xrightarrow{\theta} & \mathbb{T}
\end{array}$$

Let $\mathbb{T}$ be a pseudocommutative Lawvere 2-theory, $X_1, \ldots, X_u, Y$ are $\mathbb{T}$-models in **Cat**. Then a lax $\mathbb{T}$-multimap $X_1, \ldots, X_n \to Y$ is a lax natural transformation f



such that the precomposition with the blue square is 2-natural.

Thanks!

Connection to 2-monads

Corresponding notion of a pseudocommutativity for 2-monads[1], involving (co)strengths, 7 axioms, and the following invertible 2-cells:

$$
\begin{array}{ccccc}
TA \times TB & \xrightarrow{t^*} & T(A \times TB) & \xrightarrow{Tt} & T^2(A \times B) \\
\Big\downarrow t & & \Downarrow \gamma_{AB} & & \Big\downarrow \mu \\
T(TA \times B) & \xrightarrow[Tt^*]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B)
\end{array}
$$

If $TX = \int^n X^m \, \mathbb{T}\, m$ corresponds to a Lawvere theory $\mathbb{T}$, we have a monad $SX = \int^{m,n} X^{mn} \, \mathbb{T}\, m \times \mathbb{T}\, n$, γ_{XY} can be rewritten as

$$
TX \times TY \xrightarrow{d_{XY}} S(X \times Y) \underset{\otimes_2}{\overset{\otimes_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}}}} T(X \times Y)
$$

[1] M. Hyland, J. Power: *Pseudo-commutative monads and pseudo-closed 2-categories*, JPAA 175, p. 141-185, 2002.