

Joyal's representation theorem for Heyting categories

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Theorem (Joyal)

For any small Heyting category \mathcal{H} , there is a small category \mathbb{C} and a conservative Heyting functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$.

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A model theoretic proof has been presented by Makkai and Reyes in 1977.

Goal : to provide a categorical approach

- Posetal case : **Heyting algebras**

For any Heyting algebra \mathcal{H} , there is a poset X and an injective homomorphism of Heyting algebras $\mathcal{H} \hookrightarrow \mathcal{2}^X$

- **Stone representation theorem**

For any Boolean algebra \mathcal{B} , there is a set X and an injective Boolean homomorphism $\mathcal{B} \hookrightarrow \mathcal{2}^X$

Representation theorems \Leftrightarrow completeness theorems

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through the construction of *syntactic categories*, build from theories.

- For any coherent theory \mathbb{T} , there is a coherent category $\mathcal{C}_{\mathbb{T}}$ such that

$$\mathrm{Coh}(\mathcal{C}_{\mathbb{T}}, \mathbf{Set}) \simeq \mathrm{Mod}(\mathbb{T})$$

- A finitely complete category \mathcal{C} is *regular* if and only if :
 - any arrow in \mathcal{C} factorizes as a regular epimorphism followed by a monomorphism;
 - these factorizations are pullback-stable.
- A *coherent category* is a regular category in which posets of subobjects $\text{Sub}(a)$ have finite unions (i.e coproducts) and each pullback functor $f^{-1} : \text{Sub}(b) \rightarrow \text{Sub}(a)$ preserves them.

- A *Heyting category* is a coherent category in which for each map $f : a \rightarrow b$, the pullback functor $f^{-1} : \text{Sub}(b) \rightarrow \text{Sub}(a)$ has a right adjoint $\forall_f :$

$$\begin{array}{ccc} & f^{-1} & \\ \text{Sub}(b) & \xrightarrow{\quad} & \text{Sub}(a) \\ & \forall_f & \\ & \xleftarrow{\quad} & \end{array} \quad \perp$$

Example : Any presheaf category. In particular, $\text{PSh}(\mathbb{C}^{\text{op}}) = \mathbf{Set}^{\mathbb{C}}$

Let \mathcal{H} be a small coherent category.

Theorem

For any morphism $f : a \rightarrow b$ in \mathcal{H} , if $M(a) \cong M(b)$ for all coherent functor $M : \mathcal{H} \rightarrow \mathbf{Set}$, then f is an isomorphism.

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For any morphism $f : a \rightarrow b$ in \mathcal{H} , if $M(a) \cong M(b)$ for all coherent functors $M : \mathcal{H} \rightarrow \mathbf{Set}$, then f is an isomorphism.

\Rightarrow Gödel completeness for first-order logic

Sketch of the proof

Notations :

- $\text{Lex}(\mathcal{C}, \mathcal{D})$: category of left exact functors between finitely complete categories \mathcal{C} and \mathcal{D}
- $\text{Coh}(\mathcal{C}, \mathcal{D})$: category of coherent functors between coherent categories \mathcal{C} and \mathcal{D}

Theorem (Joyal)

For any small Heyting category \mathcal{H} , there is a small category \mathbb{C} and a conservative Heyting functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$.

For the proof, we show that \mathbb{C} can be taken to be the category of coherent functors $\mathbf{Coh}(\mathcal{H}, \mathbf{Set})$ and that the functor is given by :

$$ev : \mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$$

$$\begin{array}{ccc} a & \longmapsto & (F \mapsto F(a)) \\ \downarrow f & & \downarrow ev(f)_{F=Ff} \\ b & \longmapsto & (F \mapsto F(b)) \end{array}$$

- Conservativity of the functor : Deligne's theorem
- Coherence of the functor : $\mathbb{C} = \text{Coh}(\mathcal{H}, \mathbf{Set})$
- Preservation of the Heyting structure : we need to show that for any $f : a \rightarrow b$, and for any $u \in \text{Sub}(a)$,

$$\forall_{ev(f)}(ev(u)) = ev(\forall_f(u))$$

On objects : for any coherent functor $M \in \mathbb{C}$, we need to show that

$$\forall_{ev(f)}(ev(u))(M) = M(\forall_f(u))$$

.

- Using the definition of the universal quantification in presheaves and the definition of ev :

$$\begin{aligned} \forall_{ev(f)}(ev(u))(M) &= \{x \in M(b) \mid \text{for all } h : M \rightarrow N \text{ in } \mathbf{Coh}(\mathcal{H}, \mathbf{Set}), \\ &\quad \text{for all } y \in N(a), \text{ if } Nf(y) = h_b(x) \text{ then } y \in N(u)\} \end{aligned}$$

- Therefore, to show $\forall_{ev(f)}(ev(u))(M) \subseteq M(\forall_f(u))$, we assume $x \in M(b)$, $x \notin M(\forall_f(u))$ and we need to show that :

there exists $N \in \mathbf{Coh}(\mathcal{H}, \mathbf{Set})$, $h : M \rightarrow N$ and $y \in N(a)$ such that $Nf(y) = h_b(x)$ but $y \notin N(u)$

Sketch of the proof

We have $f : a \rightarrow b$, $u \in \text{Sub}(a)$ and $M \in \text{Coh}(\mathcal{H}, \mathbf{Set})$

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$$\begin{array}{c} yb \\ yf \downarrow \\ ya \\ \downarrow \\ yu \end{array}$$

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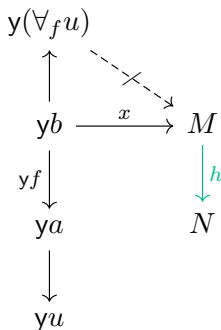
Suppose $x \in M(b)$ but $x \notin M(\forall_f u)$

$$\begin{array}{ccc} & y(\forall_f u) & \\ & \uparrow & \searrow \text{---} \times \text{---} \\ yb & \xrightarrow{x} & M \\ & \downarrow yf & \\ & ya & \\ & \downarrow & \\ & yu & \end{array}$$

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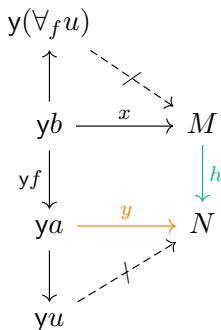


WTS : there exists $N \in \text{Coh}(\mathcal{H}, \mathbf{Set})$, $h : M \rightarrow N$

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- $\tilde{N}f(\tilde{y}) = \tilde{h}_b(x)$ by commutation of the diagram
- $\tilde{y} \notin \tilde{N}(u)$ by our assumption $x \notin M(\forall_f u)$ and the adjunction $f^{-1} \dashv \forall_f$

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Second step : A new category $\mathcal{H}(\tilde{N})$

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- $\mathcal{H}(\tilde{N})^{op} \hookrightarrow \tilde{N}/\text{Lex}(\mathcal{H}, \mathbf{Set})$ is the full subcategory
- Its objects are pushouts of maps in \mathcal{H} , e.g. :

$$\begin{array}{ccc} yd & \longrightarrow & \tilde{N} \\ \downarrow & & \downarrow \\ yc & \longrightarrow & Y \end{array} \quad \sqcap$$

for a map $c \rightarrow d$ in \mathcal{H} .

Second step : A new category $\mathcal{H}(\tilde{N})$

- $\tilde{N} \in \text{Lex}(\mathcal{H}, \mathbf{Set}) \Rightarrow \tilde{N} \cong \text{colim}_{i \in I} y\tilde{N}_i$ for I a filtered category.
- $\mathcal{H}(\tilde{N}) \simeq \text{colim}_{i \in I} \mathcal{H}/\tilde{N}_i$

Lemma

A filtered colimit of coherent categories and coherent functors between them is coherent.

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Lemma

A filtered colimit of coherent categories and coherent functors between them is coherent.

$\Rightarrow \mathcal{H}(\tilde{N})$ is coherent.

Moreover, $\text{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set}) \simeq \tilde{N} / \text{Coh}(\mathcal{H}, \mathbf{Set})$

Third step : Deligne's theorem, again

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- There is an object Y in $\mathcal{H}(\tilde{N})$:
$$\begin{array}{ccc} ya & \xrightarrow{\tilde{y}} & \tilde{N} \\ \downarrow & \lrcorner & \downarrow \\ yu & \longrightarrow & Y \end{array}$$
 such that $Y \not\cong \tilde{N}$.

Theorem (Deligne)

For any morphism $f : a \rightarrow b$ in \mathcal{H} , if $M(a) \cong M(b)$ for all $M \in \text{Coh}(\mathcal{H}, \mathbf{Set})$, then f is an isomorphism.

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- There exists $N \in \text{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set})$ such that $N(Y) \not\cong N(\tilde{N})$
- Since $\text{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set}) \simeq \tilde{N} / \text{Coh}(\mathcal{H}, \mathbf{Set})$, we have

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\quad} & N \\ \downarrow & \nearrow \text{---} & \\ Y & & \end{array}$$

Third step : Deligne's theorem, again

$$\begin{array}{ccccc} yb & \xrightarrow{x} & M & & \\ \downarrow yf & & \downarrow \tilde{h} & & \\ ya & \xrightarrow{\tilde{y}} & \tilde{N} & \xrightarrow{\quad} & N \\ \downarrow & & \downarrow & \nearrow \text{---} & \\ yu & \xrightarrow{\quad} & Y & & \end{array}$$

Diagram illustrating a commutative diagram with objects yb , M , ya , \tilde{N} , yu , Y , and N . The diagram shows maps x , \tilde{h} , \tilde{y} , yf , and yu . A blue arrow points from \tilde{N} to N . A dashed arrow points from Y to N with a cross on it. Right-angle symbols (\sqcap) are present at the corners (yb, M, ya) , (yb, M, yu) , and (ya, \tilde{N}, Y) .

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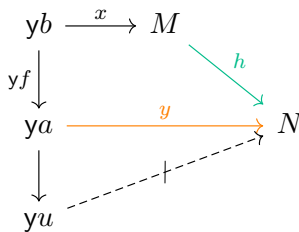
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Completeness results

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For any intuitionistic first-order theory \mathbb{T} , one can construct its *syntactic category* $\mathcal{C}_{\mathbb{T}}$.

- $\mathcal{C}_{\mathbb{T}}$ is a Heyting category.
- $\mathcal{C}_{\mathbb{T}}$ contains a *universal model* U such that a formula (of IFOL) is provable from the axioms of \mathbb{T} if and only if this formula holds in U .

$$\mathbb{T} \text{ proves } (\Gamma \mid \varphi) \quad \text{iff} \quad U \models (\Gamma \mid \varphi)$$

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Theorem (Kripke Completeness of IFOL, 1965)

If a formula of IFOL holds in every Kripke model, then it is provable in Heyting predicate calculus.

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If a formula of IFOL holds in every Kripke model, then it is provable in Heyting predicate calculus.

- Kripke semantics agree with semantics in presheaves of the form \mathbf{Set}^K , with K a poset
- Diaconescu cover : There exists a poset K and a conservative, Heyting functor $\mathbf{Set}^{\mathbb{C}} \hookrightarrow \mathbf{Set}^K$

Completeness results

Let \mathbb{T} be an IFO theory, consider Joyal's theorem for its syntactic category

$$ev : \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$$

with $\mathbb{C} = \text{Coh}(\mathcal{C}_{\mathbb{T}}, \mathbf{Set})$.

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- Since ev is a Heyting functor, the image of the universal model U in $\mathcal{C}_{\mathbb{T}}$ under ev is again a model $ev(U)$ of \mathbb{T} .

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- Since ev is a Heyting functor, the image of the universal model U in $\mathcal{C}_{\mathbb{T}}$ under ev is again a model $ev(U)$ of \mathbb{T} .
- By conservativity of ev : if a formula holds in the model $ev(U)$ it also holds in the model U , and therefore is provable.

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