

(THE SEMANTIC LAX DESCENT FACTORIZATION OF A FUNCTOR)

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109th PSSSL

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LEIDEN, THE NETHERLANDS

• AIM OF THE TALK

- LAX REGULAR IMAGE OF A FUNCTOR
(SEMANTIC LAX DESCENT FACTORIZATION)
- POINT OUT THAT THE (CO)EILENBERG MOORE
FACTORIZATION OF A (LEFT) RIGHT ADJOINT
FUNCTOR.

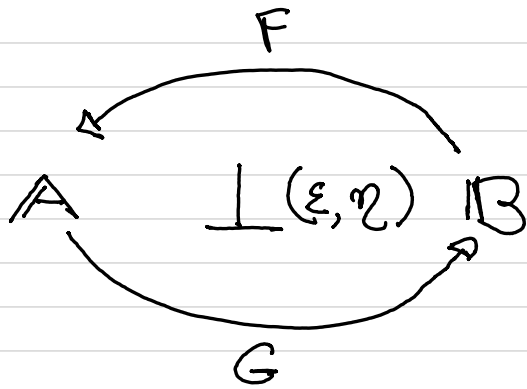
* LUCATELLI NUNES,

Semantic Factorization and Descent

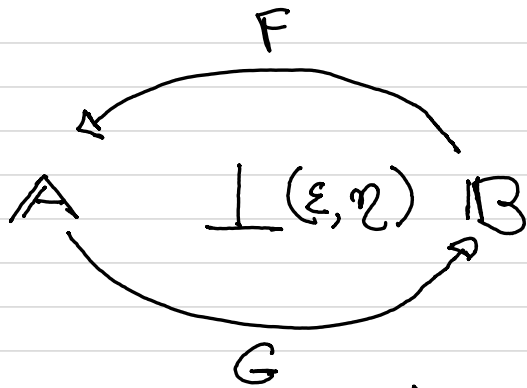
APPLIED CATEG. STRUCTURES, 2022

•

$$A \xrightarrow{G} B \quad \text{functor}$$



adjoint situation

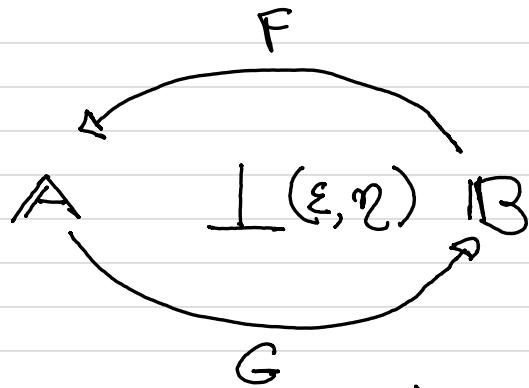


adjoint situation

→ monad $(GF, G\epsilon F, \eta)$

→ comonad $(FG, F\eta G, \epsilon)$

(P. J. Huber, 64)

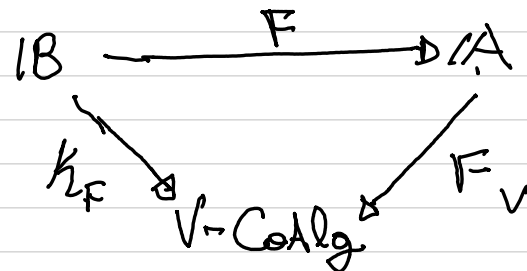
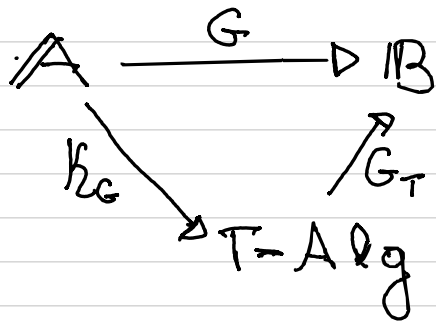


adjoint situation

→ monad $(GF, G \varepsilon F, \eta) = T$

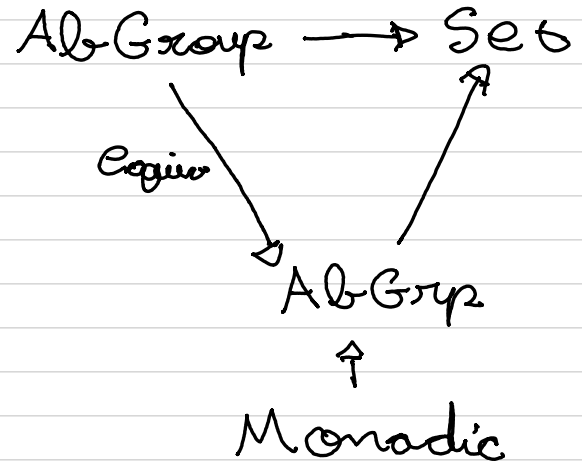
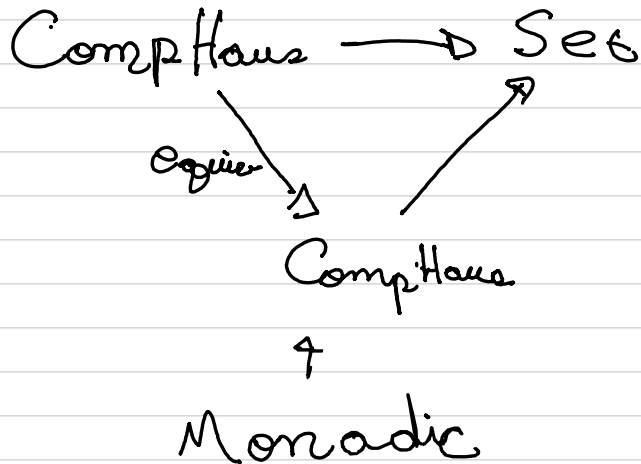
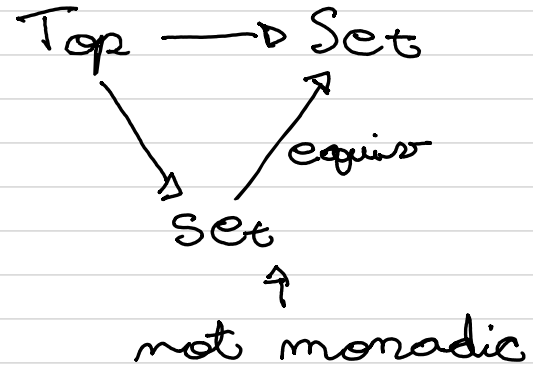
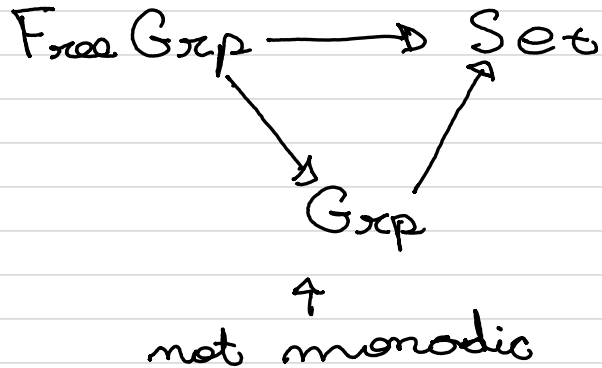
→ comonad $(FG, F \eta G, \varepsilon) = V$

• EILENBERG - MOORE FACTORIZATIONS:

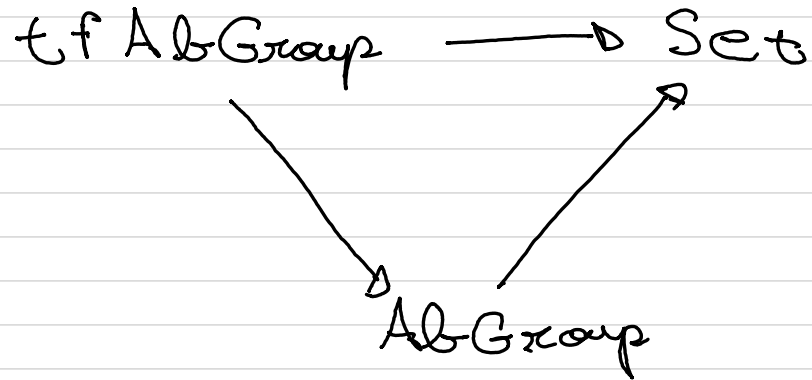


(Eilenberg and Moore, 1965)

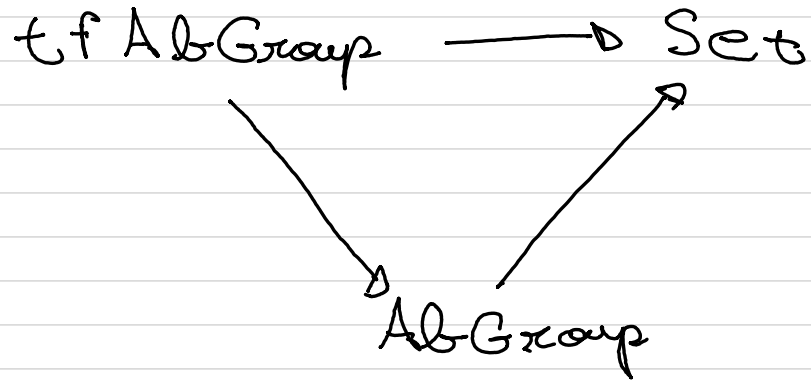
- Examples (Eilenberg - Moore Factorizations)



- Further Examples (Eilenberg - Moore Factorizations)

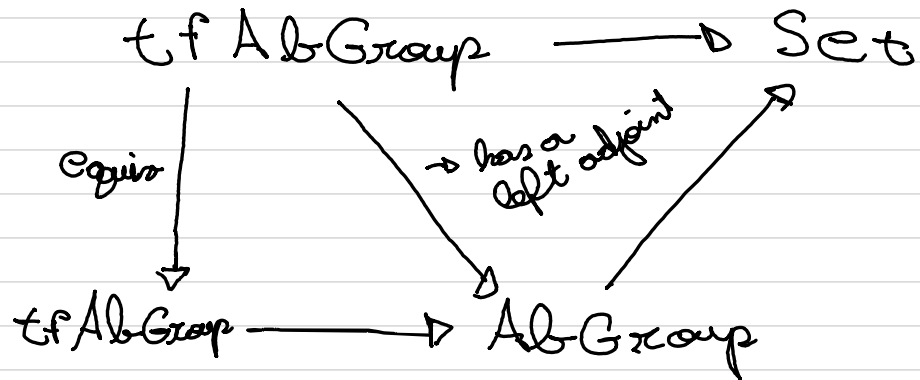


- Further Examples (Eilenberg - Moore Factorizations)

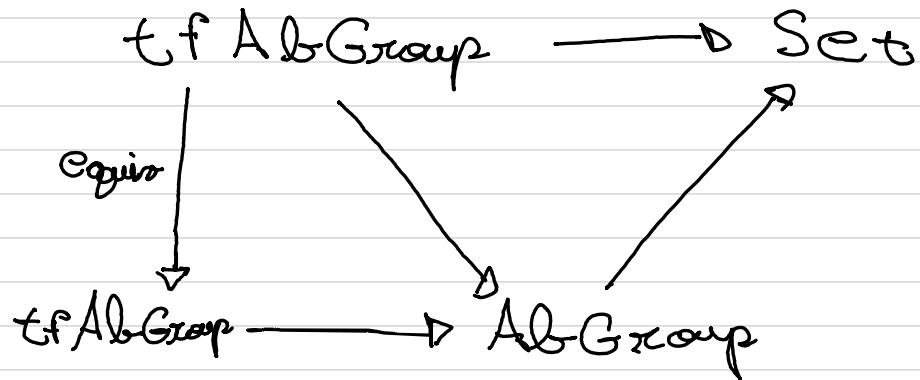


$\rightarrow$ not monadic

- Further Examples (Eilenberg - Moore Factorizations)



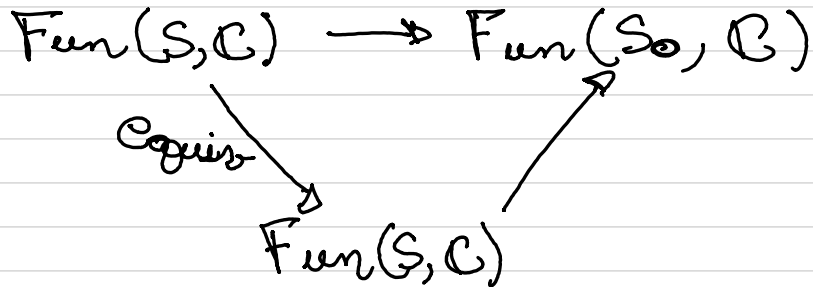
- Further Examples (Eilenberg - Moore Factorizations)



COMPOSITION OF MONADIC FUNCTORS
MIGHT NOT BE MONADIC!

- LAST EXAMPLE :

$S \rightarrow$ small category
 $\mathcal{C} \rightarrow$ cocomplete and complete



MONADIC
AND
COMONADIC

ALGEBRAS
AND
COALGEBRAS
COINCIDE.

- Getting back to the general setting:

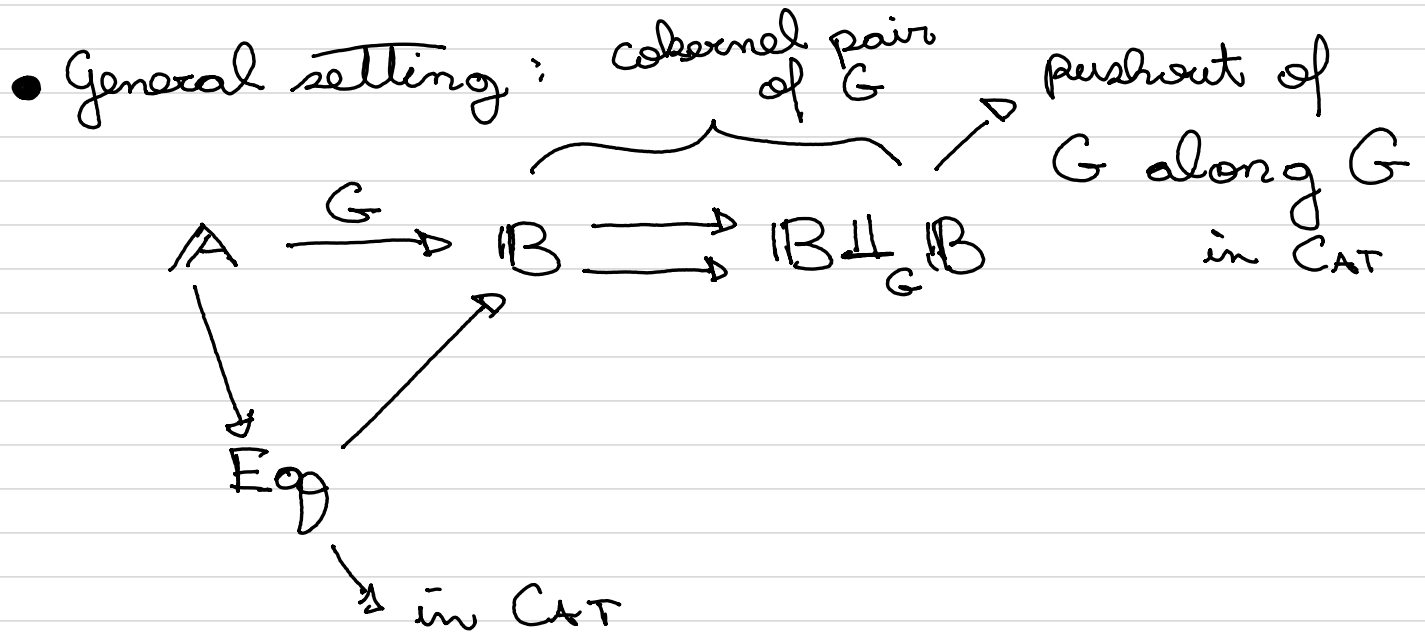
$$A \xrightarrow{G} B$$

- General setting:

$$A \xrightarrow{G} B$$

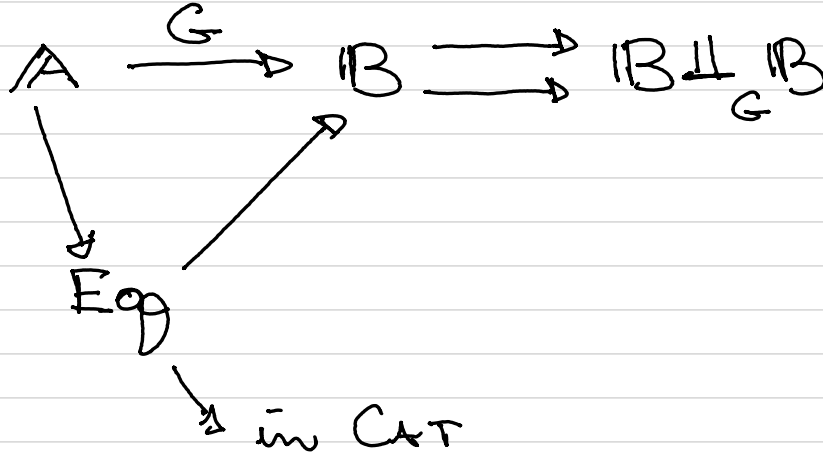
DOES NOT HAVE EILENBERG
- MOORE FACTORIZATION

- General setting: cokernel pair of G
- pushout of G along G in CAT
- $$A \xrightarrow{G} B \rightrightarrows B \amalg_G B$$



REGULAR IMAGE FACTORIZATION

- General setting: $\underbrace{\text{cokernel pair of } G}_{\text{pushout of } G \text{ along } G \text{ in CAT}}$



- CONSIDER THE 2-DIMENSIONAL STRUCTURE
- LET'S MAKE IT LAX

- General setting:

$$A \xrightarrow{G} B \rightrightarrows B \begin{matrix} \uparrow \\ G \\ \downarrow \end{matrix} B \rightrightarrows B \begin{matrix} \uparrow \\ G \\ \downarrow \end{matrix} B \begin{matrix} \uparrow \\ G \\ \downarrow \end{matrix} B$$

(Lax) TWO-DIMENSIONAL
COKERNEL DIAGRAM

- General setting:

$$A \xrightarrow{G} B \rightrightarrows B \underset{G}{\updownarrow} B \rightrightarrows B \underset{G}{\updownarrow} B \underset{G}{\updownarrow} B$$

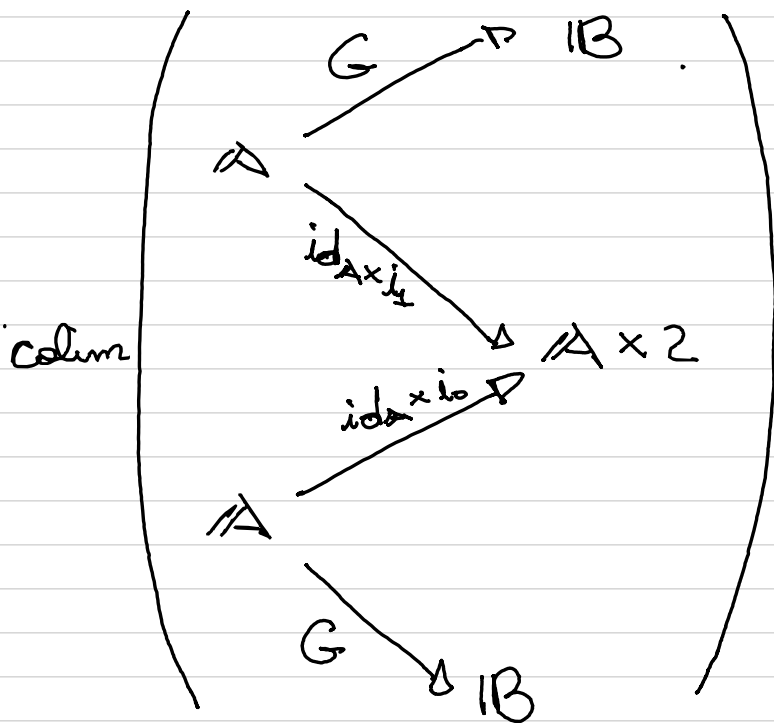
opcomma category
of G along G .

- General setting:

$$A \xrightarrow{G} B \rightrightarrows B \hat{\uparrow}_G B \rightrightarrows B \hat{\uparrow}_G B \hat{\uparrow}_G B$$

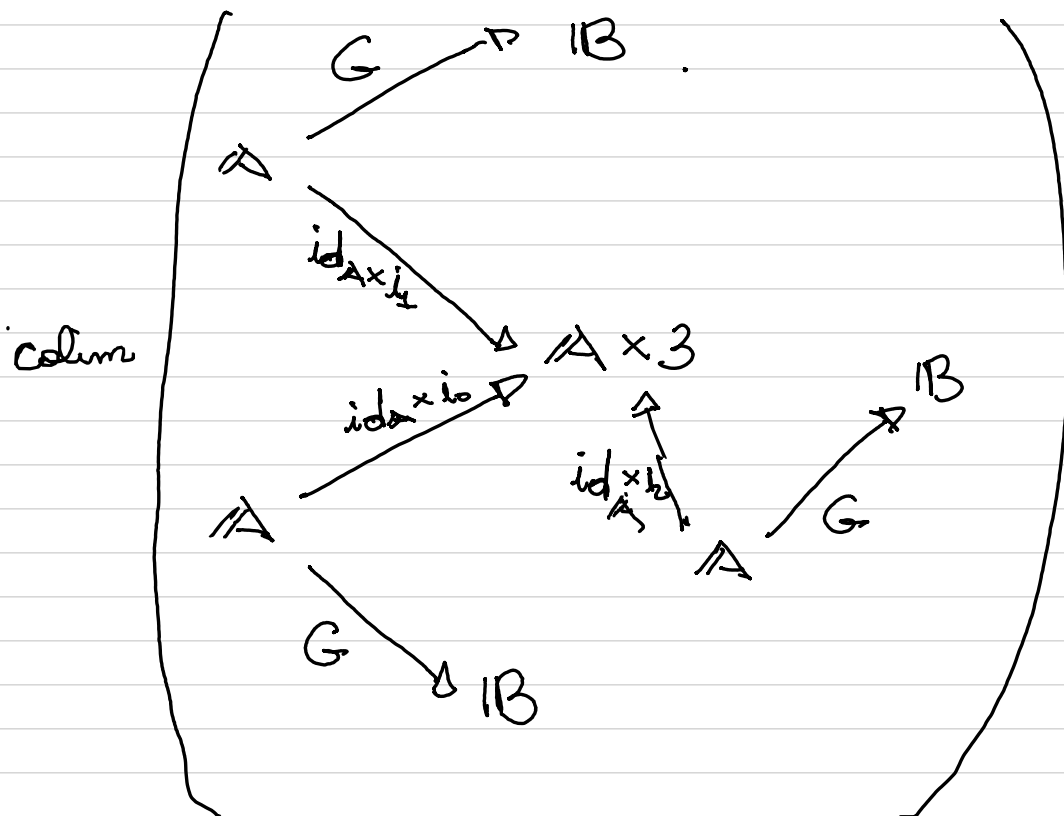
opcomma category
of G along G .

$$\cong B \hat{\uparrow}_G B$$



- General setting:

$$A \xrightarrow{G} B \rightrightarrows B \underset{G}{\overset{\uparrow}{\rightrightarrows}} B \rightrightarrows B \underset{G}{\overset{\uparrow}{\rightrightarrows}} B \underset{G}{\overset{\uparrow}{\rightrightarrows}} B$$



$$\cong B \underset{G}{\overset{\uparrow}{\rightrightarrows}} B \underset{G}{\overset{\uparrow}{\rightrightarrows}} B$$

$$3 = \begin{matrix} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 0 \end{matrix}$$

- General setting: 2-dimensional cokernel diagram of G

$$A \xrightarrow{G} B \rightrightarrows B \underset{G}{\uparrow} B \rightrightarrows B \underset{G}{\uparrow} B \underset{G}{\uparrow} B$$

- General setting:

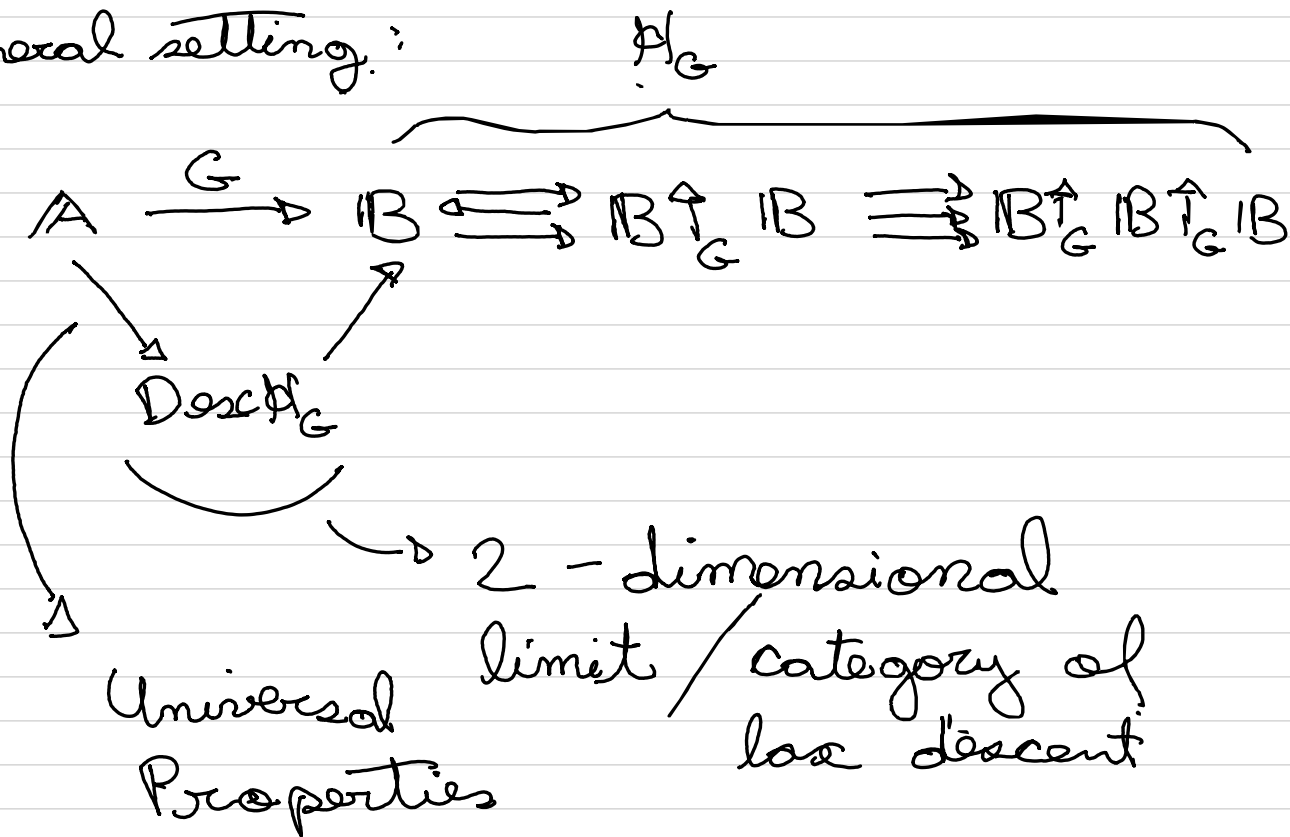
$$A \xrightarrow{G} B \xrightleftharpoons{\quad} B \uparrow_G B \xrightleftharpoons{\quad} B \uparrow_G B \uparrow_G B$$

$\mathcal{H}_G$

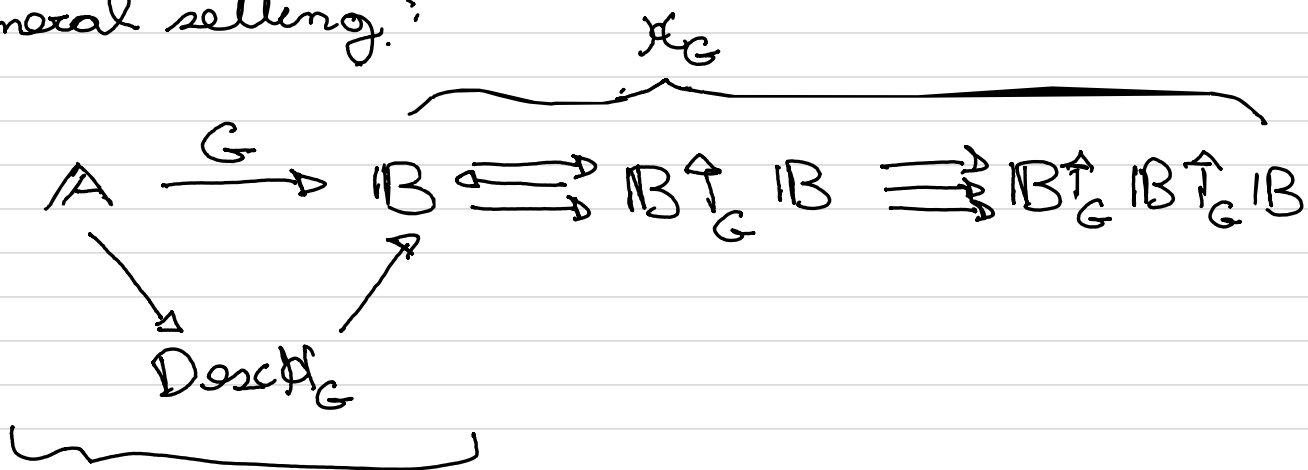
Desc $\mathcal{H}_G$

2-dimensional
limit / category of
descent data

- General setting:



- General setting:



"Semantic loss descent factorization" of G

- General setting:

$$\begin{array}{c}
 \overbrace{\quad\quad\quad}^{H_G} \\
 A \xrightarrow{G} B \rightrightarrows B \uparrow_G B \rightrightarrows B \uparrow_G B \uparrow_G B \\
 \swarrow \quad \nearrow \\
 \text{Desc } H_G
 \end{array}$$

EVERY FUNCTOR HAS A SEMANTIC
LAX DESCENT FACTORIZATION!

- General setting:

$$\begin{array}{c}
 \text{A} \xrightarrow{G} \text{B} \xrightleftharpoons{\quad} \text{B} \underset{G}{\uparrow} \text{B} \xrightleftharpoons{\quad} \text{B} \underset{G}{\uparrow} \text{B} \underset{G}{\uparrow} \text{B} \\
 \searrow \quad \circledast \quad \nearrow \\
 \text{Dox } \mathcal{H}_G
 \end{array}$$

MAIN THEOREM:

G right adjoint $\Rightarrow \circledast$ coincides w. EMF

G left adjoint $\Rightarrow \circledast$ coincides w. coEMF

Message !

THE SEMANTIC LAX DESCENT FACTORIZATION
OF ANY FUNCTOR

IS A NATURAL GENERALIZATION
OF THE

(co)EILENBERG MOORE FACTORIZATIONS!

FURTHERMORE:

➔ MAIN THEOREM HOLDS IN ANY 2-CATEGORY

(WITH SUITABLE 2-(CO)LIMITS.

➔ CAN BE SEEN AS:

① FORMAL MONADICITY THEOREM;

② A COUNTERPART TO BÉNABOU-ROBAUD THEOREM;

"DESCENT DATA IS A COMMON

GENERALIZATION OF BOTH ALGEBRAIC

AND COALGEBRAIC STRUCTURES"

Thank you

for your attention!