(The semantic lax descent factorization of a functor)

Fernando Lucatelli Nunes
Utrecht University

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• AIM OF THE TALK

- LAX REGULAR IMAGE OF A FUNCTOR (SEMANTIC LAX DESCENT FACTORIZATION)

- POINT OUT THAT THE (CO)EILENBERG MOORE FACTORIZATION OF A (LEFT) RIGHT ADJOINT FUNCTOR.

* LUCATELLI NUNES,
Semantic Factorization and Descent
APPLIED CATEG. STRUCTURES, 2022

- $\mathbb{A} \xrightarrow{G} \mathbb{B}$ functor

- $$\begin{tikzcd} \mathbb{A} \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$ adjoint situation

- $$\begin{tikzcd} \mathbb{A} \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$ adjoint situation

  $\to$ monad $(GF, G\varepsilon F, \eta)$

  $\to$ comonad $(FG, F\eta G, \varepsilon)$

(P. J. Huber, 61)

- $$\begin{tikzcd} \mathbb{A} \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] \arrow[rr, bend right, "G"'] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$ adjoint situation

  $\rightarrow$ monad $(GF, G\varepsilon F, \eta) = T$

  $\rightarrow$ comonad $(FG; F\eta G, \varepsilon) = V$

- EILENBERG-MOORE FACTORIZATIONS:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "G"] \arrow[dr, "K_G"'] & & \mathbb{B} \\ & T\text{-}Alg \arrow[ur, "G_T"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} \arrow[rr, "F"] \arrow[dr, "K_F"'] & & \mathbb{A} \arrow[dl, "F_V"] \\ & V\text{-}CoAlg & \end{tikzcd}$$

(Eilenberg and Moore, 1965)

- Examples (Eilenberg-Moore Factorization)

$$\begin{tikzcd} \mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Grp} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Top} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{set} \arrow[ur, "\text{equiv}"'] & \end{tikzcd}$$

not monadic — not monadic

$$\begin{tikzcd} \mathrm{CompHaus} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{CompHaus} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{AbGroup} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{AbGrp} \arrow[ur] & \end{tikzcd}$$

Monadic — Monadic

- Further Examples (Eilenberg - Moore Factorization)

$$\begin{tikzcd}
\mathrm{tfAbGroup} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\
& \mathrm{AbGroup} \arrow[ur] &
\end{tikzcd}$$

- Further Examples (Eilenberg - Moore Factorization)

$$\begin{tikzcd}
\text{tf AbGroup} \arrow[rr] \arrow[dr] & & \text{Set} \\
& \text{AbGroup} \arrow[ur] &
\end{tikzcd}$$

$\rightarrow$ not monadic

- Further Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd}
\text{tfAbGroup} \arrow[rr] \arrow[d, "\text{equiv}"'] \arrow[dr, "\text{has a left adjoint}"] & & \text{Set} \\
\text{tfAbGroup} \arrow[r] & \text{AbGroup} \arrow[ur] &
\end{tikzcd}$$

- Further Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd} \mathrm{tfAbGroup} \arrow[rr] \arrow[d, "\text{equiv}"'] \arrow[dr] & & \mathrm{Set} \\ \mathrm{tfAbGroup} \arrow[r] & \mathrm{AbGroup} \arrow[ur] & \end{tikzcd}$$

COMPOSITION OF MONADIC FUNCTORS MIGHT NOT BE MONADIC!

• LAST EXAMPLE:

$S \to$ small category
$\mathbb{C} \to$ Cocomplete and complete

$$
\begin{tikzcd}
\mathrm{Fun}(S,\mathbb{C}) \arrow[rr] \arrow[dr, "\text{Equiv}"'] & & \mathrm{Fun}(S_0,\mathbb{C}) \\
& \mathrm{Fun}(S,\mathbb{C}) \arrow[ur] &
\end{tikzcd}
$$

MONADIC

AND

COMONADIC

ALGEBRAS

AND

COALGEBRAS

COINCIDE.

- Getting back to the general setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B}$$ DOES NOT HAVE EILENBERG-MOORE FACTORIZATION

- General setting: cokernel pair of G $\to$ pushout of G along G in CAT

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \rightrightarrows \mathbb{B} \amalg_G \mathbb{B}$$

- General setting:

cokernel pair of G

pushout of G along G in CAT

$$\begin{tikzcd} \mathbb{A} \arrow[r, "G"] \arrow[d] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \amalg_G \mathbb{B} \\ \mathrm{Eq} \arrow[ur] & & \end{tikzcd}$$

in CAT

REGULAR IMAGE FACTORIZATION

• General setting:

cokernel pair of G

pushout of G along G in CAT

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \amalg_G \mathbb{B} \\
\mathrm{Eq} \arrow[ur] &
\end{tikzcd}
$$

in CAT

• CONSIDER THE 2-DIMENSIONAL STRUCTURE

• LET'S MAKE IT LAX

- General setting:

$$\mathbb{A} \xrightarrow{G} \underbrace{\mathbb{B} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathbb{B} \uparrow_G \mathbb{B} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}}_{\substack{\text{(LAX) TWO-DIMENSIONAL} \\ \text{COKERNEL DIAGRAM}}}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \underbrace{\mathbb{B} \uparrow_G \mathbb{B}}_{\text{opcomma category of } G \text{ along } G.} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \rightleftarrows \underbrace{\mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}}_{\text{opcomma category of } G \text{ along } G.} \rightrightarrows \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}$$

$$\operatorname{colim}\left(\begin{tikzcd} & \mathbb{B} \\ \mathbb{A} \arrow[ur, "G"] \arrow[dr, "id_{\mathbb{A}} \times i_1"'] & \\ & \mathbb{A} \times 2 \\ \mathbb{A} \arrow[ur, "id_{\mathbb{A}} \times i_0"] \arrow[dr, "G"'] & \\ & \mathbb{B} \end{tikzcd}\right) \simeq \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \rightleftarrows \mathbb{B} \uparrow_G \mathbb{B} \Rrightarrow \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}$$

$$\mathrm{colim}\left(\begin{tikzcd}
 & \mathbb{B} & & \\
\mathbb{A} \arrow[ur, "G"] \arrow[dr, "id_A \times i_1"'] & & & \\
 & \mathbb{A} \times 3 & & \mathbb{B} \\
\mathbb{A} \arrow[ur, "id_A \times i_0"] \arrow[ddr, "G"'] & & & \\
 & & \mathbb{A} \arrow[uul, "id_A \times i_2"'] \arrow[uur, "G"'] & \\
 & \mathbb{B} & &
\end{tikzcd}\right) \cong \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}$$

$$3 = \begin{tikzcd} 2 \\ 1 \arrow[u] \\ 0 \arrow[u] \end{tikzcd}$$

• General setting: 2-dimensional cokernel diagram of G

$$\mathbb{A} \xrightarrow{G} \underbrace{\mathbb{B} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathbb{B} \mathbin{\uparrow_G} \mathbb{B} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \mathbb{B} \mathbin{\uparrow_G} \mathbb{B} \mathbin{\uparrow_G} \mathbb{B}}$$

- General setting:

$$\begin{tikzcd} & & \mathcal{H}_G & \\ A \arrow[r, "G"] & \mathbb{B} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{B}\uparrow_G \mathbb{B} \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \mathbb{B}\uparrow_G \mathbb{B}\uparrow_G \mathbb{B} \\ \mathrm{Desc}\,\mathcal{H}_G \arrow[ur] & & & \end{tikzcd}$$

2-dimensional limit / category of descent data

- General setting:

$$A \xrightarrow{G} \overbrace{\mathbb{B} \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow}} \mathbb{B}\uparrow_G \mathbb{B} \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} \mathbb{B}\uparrow_G \mathbb{B}\uparrow_G \mathbb{B}}^{\mathcal{A}_G}$$

$$A \longrightarrow \mathrm{Desc}\,\mathcal{A}_G \longrightarrow \mathbb{B}$$

Universal Properties

2-dimensional limit / category of lax descent

- General setting:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{B}\uparrow_G\mathbb{B} \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \mathbb{B}\uparrow_G\mathbb{B}\uparrow_G\mathbb{B} \\
& \mathrm{Desc}\,\mathcal{X}_G \arrow[ur] & & &
\end{tikzcd}$$

$\mathcal{X}_G$

"Semantic lax descent factorization" of G

- General setting:

$$\begin{tikzcd}
A \arrow[r, "G"] \arrow[d] & \overbrace{\mathbb{B} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbb{B}\uparrow_G \mathbb{B} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbb{B}\uparrow_G \mathbb{B}\uparrow_G \mathbb{B}}^{\mathcal{H}_G} \\
\mathrm{Desc}\,\mathcal{H}_G \arrow[ur]
\end{tikzcd}$$

EVERY FUNCTOR HAS A SEMANTIC LAX DESCENT FACTORIZATION!

- General setting:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "G"] \arrow[dr] & \arrow[d, phantom, "\circledast"] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B}\uparrow_G \mathbb{B} \arrow[l] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \mathbb{B}\uparrow_G \mathbb{B}\uparrow_G \mathbb{B} \\ & \mathrm{Desc}\,\mathbb{H}_G \arrow[ur] & & & \end{tikzcd}$$

where $\mathbb{H}_G$ denotes $\underbrace{\mathbb{B} \rightleftarrows \mathbb{B}\uparrow_G \mathbb{B} \ \substack{\to\\ \to \\ \to}\ \mathbb{B}\uparrow_G \mathbb{B}\uparrow_G \mathbb{B}}$

MAIN THEOREM:

G right adjoint $\Longrightarrow$ $\circledast$ coincides w. EMF

G left adjoint $\Longrightarrow$ $\circledast$ coincides w. co EMF

Message:

THE SEMANTIC LAX DESCENT FACTORIZATION OF ANY FUNCTOR:

IS A NATURAL GENERALIZATION OF THE

(co)EILENBERG MOORE FACTORIZATIONS!

FURTHERMORE:

- MAIN THEOREM HOLDS IN ANY 2-CATEGORY (WITH SUITABLE 2-(CO)LIMITS.
- CAN BE SEEN AS:

  (A) FORMAL MONADICITY THEOREM;

  (B) A COUNTERPART TO BÉNABOU-ROBAUD THEOREM;

  "DESCENT DATA IS A COMMON GENERALIZATION OF BOTH ALGEBRAIC AND CO ALGEBRAIC STRUCTURES"

Thank you

for your attention!