Strictifying monoidal structure, revisited

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Outline

Two kinds of strictification

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- Strictifying a monoidal category means finding an equivalent monoidal category that's strict.
- This is always possible.
- Let $\mathcal{C}$ be a category. Strictifying a monoidal structure on $\mathcal{C}$ means finding an isomorphic monoidal structure on $\mathcal{C}$ that's strict.
- When is this possible?

For structured sets, structure strictification isn't straightforward

- Schauenburg (2001) claimed that, on a category of structured sets, any monoidal structure can be strictified.
- This was incorrect, but his methods work in many cases.

Counterexample

Let $\mathcal{C}$ be the category of sets with at most one element, and bijections.
Monoidal structure:

$$\begin{aligned} \emptyset \oplus \emptyset &\stackrel{\text{def}}{=} \emptyset \\ \emptyset \oplus \{x\} &\stackrel{\text{def}}{=} \{0\} \\ \{x\} \oplus \emptyset &\stackrel{\text{def}}{=} \{0\} \\ \{x\} \oplus \{y\} &\stackrel{\text{def}}{=} \emptyset \end{aligned}$$

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A strictification $\square$ would satisfy

$$\begin{aligned} (\{0\}\square\{0\})\square\{1\} &= \{0\}\square(\{0\}\square\{1\}) \\ \emptyset\square\{1\} &= \{0\}\square\emptyset \\ \{1\} &= \{0\} \end{aligned}$$

Label-bearing categories

A label-bearing category consists of the following:

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1. A category $\mathcal{C}$, with certain objects designated empty.
   Requirement
   Any isomorphism to or from an empty object is an identity.
2. For any thing x and object a, an isomorph $(x \cdot a,\ \theta_{a,x})$ of a.
   Requirement
   If a is nonempty, then $x \cdot a = y \cdot b$ implies $x = y$.

Examples of label-bearing categories

- $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Top}$. Take $x \cdot a \stackrel{\text{def}}{=} \{x\} \times a$.
- Categories of structured sets.
- $\mathcal{C}^{\mathsf{op}}$, for a label-bearing category $\mathcal{C}$.
- $\prod_{i \in I} \mathcal{C}_i$, for label-bearing categories $(\mathcal{C}_i)_{i \in I}$.
  An object is empty when all its components are.
- $\mathsf{Fam}(\mathcal{C})$, for a category $\mathcal{C}$.

Results

Key result

On a label-bearing category,
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Compositionality

- Product-like structure on $\mathcal{C}$ gives one on $\mathcal{C}^{\mathsf{op}}$.
- Sum-like structure on $\mathcal{C}$ gives one on $\mathcal{C}^{\mathsf{op}}$.
- Product-like structures on $(\mathcal{C}_i)_{i \in I}$ gives one on $\prod_{i \in I} \mathcal{C}_i$.
- Sum-like structures on $(\mathcal{C}_i)_{i \in I}$ gives one on $\prod_{i \in I} \mathcal{C}_i$.

Product-like structure

On a label-bearing category $\mathcal{C}$, a monoidal structure is product-like when $a \otimes b$ is empty if a or b is.

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- Why not require that $a \otimes b$ is nonempty if a and b are?
  Because this would exclude the product on $\mathbf{Set}^2$.
- Why not require the unit to be nonempty?
  Because this would exclude the product on $\mathbf{Set}^0$.

Decomposing the unit

A monoid has indecomposable unit when
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Examples

1. Nonnegative reals under addition.
2. Any monoid where $\otimes$ is idempotent. Proof:

$$\begin{array}{rcl} a \otimes b \text{ implies } a & = & a \otimes I \\ & = & a \otimes a \otimes b \\ & = & a \otimes b \\ & = & I \end{array}$$

Sum-like structure

On a label-bearing category $\mathcal{C}$, a monoidal structure is sum-like when:

- The unit is empty.
- $a \otimes b$ is empty iff both a and b are.
- The class monoid of empty objects has indecomposable unit.

All monoidal structures?

Let $\mathcal{C}$ be a label-bearing category such that:

- Every nonempty object is weakly terminal.
- Every empty object is initial.
- Every morphism to an empty object is from an empty object.

Then every monoidal structure on $\mathcal{C}$ can be strictified.

Examples include $\mathbf{Set}$ and $\mathbf{Poset}$.

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Examples include $\mathbf{Set}$ and $\mathbf{Poset}$.

Don't know about $\mathbf{Rel}$ or $\mathbf{Bij}$ or $\mathbf{Set}^2$ or $\mathrm{Fam}(\mathbf{Set})$.

Warning: preservation of inclusions

On $\mathbf{Set}$, the standard implementation of $\times$ and $+$ preserve inclusions.

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On **Set**, the standard implementation of $\times$ and $+$ preserve inclusions.

But no implementation of products preserves inclusions and is strict.

Likewise for sums.

Conclusions

Many categories such as $\mathbf{Set}$ and $\mathbf{Set}^2$ and $\mathsf{Fam}(\mathbf{Set})$ are label-bearing.

On such a category, a monoidal structure that is either product-like or sum-like can be strictified.

And for certain categories e.g. $\mathbf{Set}$, this means any monoidal structure.

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Bonus corollary

As a "category with two monoidal structures",

$(\mathbf{Set}, \times, +)$ is equivalent to one where both structures are strict.