Pretorsion Theories on Quasi-catégories

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Our plan and similar work in the literature

In this talk, we will discuss a generalization of the notion of *pretorsion theory* to the context of infinity categories (here quasi-catégories). Many crucial ideas, including the following, have already been discussed in the literature.

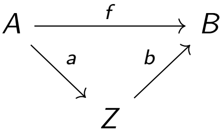
- The concept of *pretorsion theory* as a generalization of Dickson's *torsion theories* (see [1]) has been developed extensively for 1-categories by Facchini, Finocchiaro, Gran, and others. See, for instance, [2] and [3].
- There is a notion of torsion theory for stable $(\infty, 1)-$categories introduced by Fiorenza and Loregian in [4] using t-structures.

Fundamental Definitions - **Z**-triviality

The ingredients for classical pretorsion theories:
Let $\mathbf{C}$ be a category, $\mathbf{Z} \subseteq \mathbf{C}$ a subcategory, and $f : A \to B$ a morphism in $\mathbf{C}$.

Definition

f is $\mathbf{Z}-$trivial if it factors through an object $z \in \mathbf{Z}$. In otherwords, if we have the following commutative diagram:



,

i.e. $f \cong b \circ a$.

Fundamental Definitions - $\mathbf{Z}$-(co)kernels

Definition

([2], [3]) Let $\mathbf{C}$ be a category, $\mathbf{Z} \subseteq \mathbf{C}$ a subcategory, and $f : A \to A'$ a morphism in $\mathbf{C}$. The morphism $\epsilon : X \to A$ is a $\mathbf{Z}-$*kernel* of f if the following properties hold:

1. The composition $f\epsilon$ is a $\mathbf{Z}-$trivial morphism.
2. Every time that $\lambda : Y \to A$ is a morphism in $\mathbf{C}$ and the composition $f\lambda$ is $\mathbf{Z}-$trivial, there exists a unique morphism $\lambda' : Y \to X$ in $\mathbf{C}$ such that $\lambda = \epsilon\lambda'$.

Note:

- To obtain a $\mathbf{Z}-$cokernel, one dualizes the above definition.
- If $\mathbf{Z}$ is $\varnothing$, one returns to the classical definition of (co)kernel.
- There is no guarantee that such (co)kernels exist in a given category.

Fundamental Definitions: Short **Z**-exact sequence

Definition

[3], [2] A *short* $\mathbf{Z}-$*exact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in a category $\mathbf{C}$ is a pair of morphisms $f : A \to B$ and $g : B \to C$ such that f is a $\mathbf{Z}-$kernel of g and g is a $\mathbf{Z}-$kernel of f

Pretorsion theories

Definition

(Definition 2.6 of [3]) Let $\mathbf{C}$ be a category. A *pretorsion theory* $(\mathbf{T}, \mathbf{F})$ on $\mathbf{C}$ consists of a pair of full replete subcategories $\mathbf{T}$ and $\mathbf{F}$ such that for $\mathbf{Z} := \mathbf{T} \cap \mathbf{F}$, the following conditions are satisfied:

1. $hom_{\mathbf{C}}(T, F) = Triv_{\mathbf{Z}}(T, F)$ for every object $T \in \mathbf{T}$ and $F \in \mathbf{F}$.
2. For each object $B \in \mathbf{C}$ there exists a short $\mathbf{Z}-$exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $A \in \mathbf{T}$ and $C \in \mathbf{F}$.

Remark

- If $\mathbf{Z} = \varnothing$, this structure is that of a *torsion theory* (see [1]).

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Version quasi-catégorique: Fundamental Definitions

Definition

Let $\mathbf{C}$ be a quasi-catégorie and $\mathbf{Z} \subset \mathbf{C}$ a subcategory. A morphism $f : A \to B$ in $\mathbf{C}$ is $\mathbf{Z}-$*trivial* if there exists, for at least one $z \in \mathbf{Z}$ two morphismes $a \in \mathbf{C}_{/z}$ with source A and $b \in \mathbf{C}_{z/}$ with target B such that $f \cong b \circ a$.

Verification

Under the taking of the homotopy category $h\mathbf{C}$ of $\mathbf{C}$, one recovers there the $1-$categorical version of $\mathbf{Z}-$triviality.

Remark

One may also "go the other way" in a certain sense: with an additional assumption, one can show that $h\mathbf{Z}$ triviality on $h\mathbf{C}$ translates to that on $\mathbf{C}$.

(Proof idea)

Proof:

Starting in $\mathbf{C}$, we have that $f \simeq b \circ a$. To put it in a different way, there is at least one homotopy ($2-$morphism), $\aleph$, up to higher homotopy, which connects f and $b \circ a$. Under the taking of the homotopy category, then, f and $b \circ a$ are identified (the morphisms in $h\mathbf{C}$ are the homotopy classes of the morphisms in $\mathbf{C}$). In $\mathbf{C}$, a $1-$category, one has thus that f is also $h\mathbf{Z}-$trivial in the $1-$categorical sense in that $f = b \circ a$ (since they are identified, and thus homotopic).

QCs: **Z**-(co)kernels

Definition

Let **C** be an $(\infty, 1)-$category, $\mathbf{Z} \subseteq \mathbf{C}$ a subcategory, z and z' objects of **Z**, and $g : A \to B$ a morphism in **C**. The **Z**−*kernel* of g is the pullback $(\infty, 1)$

$$\begin{array}{ccc} ker(g) & \xrightarrow{\eta} & B \\ \vdots & & \downarrow g \\ z & \cdots\cdots\to & C \end{array} .$$

Dually, the **Z**−*conoyau* of a morphism $f : B \to C$ dans **C** is defined by a pushout diagram.

QCs: Short **Z**-exact sequences

Definition

Let **C** be a quasi-catégorie and $\mathbf{Z} \subseteq \mathbf{C}$ a subcategory thereof. A *short* **Z**$-$*exact sequence*

$$A \xrightarrow{\epsilon} B \xrightarrow{\eta} C,$$

consists of two morphisms $\epsilon : A \to B$ and $\eta : B \to C$ such that ϵ is a **Z**$-$kernel of η and η is a **Z**$-$cokernel of ϵ.

QCs: Pretorsion Theories

Definition

A *pretorsion theory* on a quasi-catégorie $\mathbf{C}$ consists of a triple $(\mathbf{T}, \mathbf{F}, \mathbf{Z})$ of full, replete subcategories of $\mathbf{C}$ such that the following conditions are fulfilled:

1. $Hom_{\mathbf{C}}(T, F) = Triv_{\mathbf{Z}}(T, F)$ for $T \in \mathbf{T}$ and $F \in \mathbf{F}$.
2. For every object $B \in \mathbf{C}$ there exists a short $\mathbf{Z}-$exact sequence

$$T \xrightarrow{\epsilon} B \xrightarrow{\eta} F$$

   with $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

- Here we consider $\mathbf{Z}$ as any full, replete subcategory of $\mathbf{C}$, a generalization also interesting in the $1-$categorical case. To return to the classical case, take $\mathbf{Z} = \mathbf{T} \cap \mathbf{F}$.
- One may show that this structure passes to a $1-$categorical pretorsion theory, as it were, on $h\mathbf{C}$.

QCs: Properties of Pretorsion Theories 1

Definition

(Generalization of Definition 4.1 of [2]) Let $\mathbf{C}$ be a quasi-catégorie and $\mathbf{Z} \subseteq \mathbf{C}$ a full, nonempty subcategory thereof. A full replete subcategory $\mathbf{S} \subseteq \mathbf{C}$ is *closed under extension* by $\mathbf{Z}$ if, for every short $\mathbf{Z}-$exact sequence $S_1 \to X \to S_2$ in $\mathbf{C}$ such that for S_1, $S_2 \in \mathbf{S}$ and X any object in $\mathbf{C}$, one has $X \in \mathbf{S}$.

Proposition

(Generalization of proposition 4.2 of [2].) Let $(\mathbf{T}, \mathbf{F}, \mathbf{Z})$ be a pretorsion theory on a quasi-catégorie $\mathbf{C}$ such that $\mathbf{Z} := \mathbf{T} \cap \mathbf{F}$. Then, $\mathbf{T}$, $\mathbf{F}$, and $\mathbf{Z}$ are all closed under extensions by $\mathbf{Z}$.

QCs: Properties of Pretorsion Theories 2

Proposition

(Generalization of [3] Propositions 2.1 et 2.2) Let $\mathbf{C}$ be a quasi-catégorie and $\mathbf{Z} \subseteq \mathbf{C}$ a subcategory thereof.

- $\mathbf{Z}-$kernels of the same morphism are homotopic.
- $\mathbf{Z}-$cokernels of the same morphism are homotopic.

Proposition

Let $\mathbf{C}$ be a quasi-catégorie and $(\mathbf{T}, \mathbf{F}, \mathbf{Z})$ a pretorsion theory thereupon. Then $\mathbf{F}$ is a reflective subcategory of $\mathbf{C}$ and $\mathbf{T}$ is a coreflective subcategory of $\mathbf{C}$.

Examples

We consider here one family of examples, coming from a 1−categorical construction of pretorsion theories by Cafaggi in [6].

Example: Simplicial Groups

We consider the category of simplicial groups. A simplicial group can be viewed as a Kan complex (through a specific algorithm), or an infinity groupoid. Reinterpreting, then, we consider a family of examples of pretorsion theories on $\infty - Grpd$, the $(\infty, 1)-$category of $\infty-$groupoids.

- **T** is the category of simplicial groups with associated Moore complex that is trivial above a certain degree n.
- **F** is the category of simplicial groups with associated Moore complex trivial below a certain degree m.
- If we take $\mathbf{Z} = \mathbf{T} \cap \mathbf{F}$, then, (which is not actually necessary here), it contains simplicial groups that have non-trivial Moore complex only for a finite number of degrees (between n and m).

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Future Work and Generalizations

Generalizations

1. Pretorsion theories for $n-$categories.
2. *Pointed torsion theories*: Torsion theories, but such that every morphism between $\mathbf{T}$ and $\mathbf{F}$ factorizes through one particular object).
3. *Ideal pretorsion theories*: Pretorsion theories, but where each object has a morphism to an object $z \in \mathbf{Z}$ and from an object $z' \in \mathbf{Z}$, and it is not necessary that $z = z'$.

Future Work

1. Concoct more examples of PTTS sur QCs.
2. Construction of pointed torsion theories from pretorsion theories.
3. Explore these and other possible generalizations further.

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