Monad theory through enrichment

Soichiro Fujii
Masaryk University
(Joint work with John Bourke)

$\mathcal{C}$: category

$T : \mathcal{C} \longrightarrow \mathcal{C}$: monad

$$\Longrightarrow \quad \begin{tikzcd} \overset{\text{Kleisli}}{\mathcal{C}_T} \arrow[rr, "K"] \arrow[dr, "U_T"'] & & \overset{\text{Eilenberg-Moore}}{\mathcal{C}^T} \arrow[dl, "U^T"] \\ & \mathcal{C} & \end{tikzcd}$$

$\mathcal{C}_T \simeq$ full subcat. of $\mathcal{C}^T$ consisting of the free T-algebras $\left(TA, \begin{array}{c} T^2A \\ \downarrow \mu_A \\ TA \end{array}\right)$.

Every T-algebra $\left(A, \begin{array}{c} TA \\ \downarrow \alpha \\ A \end{array}\right)$ is the canonical coequalizer of free T-algebras:

$$\left(T^2A, \begin{array}{c} T^3A \\ \downarrow \mu_{TA} \\ T^2A \end{array}\right) \underset{T\alpha}{\overset{\mu_A}{\rightrightarrows}} \left(TA, \begin{array}{c} T^2A \\ \downarrow \mu_A \\ TA \end{array}\right) \overset{\alpha}{\twoheadrightarrow} \left(A, \begin{array}{c} TA \\ \downarrow \alpha \\ A \end{array}\right) \quad : \text{coeq. in } \mathcal{C}^T.$$

$\boxed{\text{Q1.}}$ Is $\mathcal{C}^T$ a "(co)completion" of $\mathcal{C}_T$ in a suitable sense?

$\boxed{\text{A1.}}$ When e.g. $\mathcal{C} = \mathbf{Set}$, $\mathcal{C}^T$ is the exact completion of the weakly left exact category $\mathcal{C}_T$. [Vitale, Left covering functors, 1994]

What about general $\mathcal{C}$?

In general, $\mathcal{C}^T$ might not be exact, might not have (reflexive) coequalizers, ... In fact, $\mathcal{C}^T$ could be <u>any</u> category!

However, $\mathcal{C}^T$ has certain (co)completeness relative to $\mathcal{C}$.

- $\mathcal{C}^T$ lifts limits in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "X"] & \mathcal{C}^T \arrow[d, "U^T"] \\ & \mathcal{C} \end{tikzcd}$$

If $\lim U^T X$ exists in $\mathcal{C}$, then $\lim X$ exists and $U^T(\lim X) \cong \lim U^T X$.
(U^T creates limits.)

- $\mathcal{C}^T$ lifts colimits in $\mathcal{C}$ preserved by T and T^2.
- In particular, $\mathcal{C}^T$ lifts absolute colimits in $\mathcal{C}$.

The coequalizer

$$\left(T^2A, \begin{matrix} T^3A \\ \downarrow \mu_{TA} \\ T^2A \end{matrix}\right) \overset{\mu_A}{\underset{T\alpha}{\rightrightarrows}} \left(TA, \begin{matrix} T^2A \\ \downarrow \mu_A \\ TA \end{matrix}\right) \xrightarrow{\alpha} \left(A, \begin{matrix} TA \\ \downarrow \alpha \\ A \end{matrix}\right)$$

is a lifting of an absolute (in fact, split) coequalizer in $\mathcal{C}$:

$$\begin{tikzcd} T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\alpha"'] & TA \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta_{TA}"] & A \arrow[l, bend left=60, "\eta_A"] \end{tikzcd}$$

$\Longrightarrow$ Instead of the ordinary category theory, let us work in the category theory over $\mathcal{C}$.

Eilenberg-Moore $\qquad\qquad$ Kleisli

$\boxed{\text{Q2.}}$ Is $\begin{matrix} \mathcal{C}^T \\ \downarrow U^T \\ \mathcal{C} \end{matrix}$ a (co)completion of $\begin{matrix} \mathcal{C}_T \\ \downarrow U_T \\ \mathcal{C} \end{matrix}$ in a suitable sense?

... but what exactly is "category theory over $\mathcal{C}$"?

We take the following view:

(weighted) (co)limits, free (co)completion, ...

category theory over $\mathcal{C}$ = enriched category theory over a bicategory $\Sigma\mathrm{Set} \downarrow \mathcal{C}$.

[Fujii and Lack, The oplax limit of an enriched category, 2024]

$$\mathrm{Cat}/\mathcal{C} \cong (\Sigma\mathrm{Set} \downarrow \mathcal{C})\text{-}\mathrm{Cat}.$$

2-categories

Bicategory-enriched category theory [Betti, Carboni, Street, Walters, ...]

$\mathcal{B}$: bicategory

A $\mathcal{B}$-category $\mathbb{A}$ consists of:

- a set $\mathrm{ob}(\mathbb{A})$ of objects equipped with a function

$$\begin{array}{ccc} \mathrm{ob}(\mathbb{A}) & \xrightarrow{|\cdot|} & \mathrm{ob}(\mathcal{B}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & |x| \end{array}$$

$|x|$: the extent of x.

- $\forall x, y \in \mathrm{ob}(\mathbb{A})$. a 1-cell $|x| \xrightarrow{\mathbb{A}(x,y)} |y|$ in $\mathcal{B}$.

the hom 1-cell from x to y.

- $\forall x \in \mathrm{ob}(\mathbb{A})$. a 2-cell
$$\begin{tikzcd} {|x|} \arrow[r, bend left, "I_{|x|}"] \arrow[r, bend right, "\mathbb{A}(x,x)"'] \arrow[r, phantom, "\Downarrow j_x"] & {|x|} \end{tikzcd}$$
in $\mathcal{B}$.

- $\forall x, y, z \in \mathrm{ob}(\mathbb{A})$. a 2-cell
$$\begin{tikzcd} & {|y|} \arrow[dr, "\mathbb{A}(y,z)"] \arrow[d, phantom, "\Downarrow m_{xyz}"] & \\ {|x|} \arrow[ur, "\mathbb{A}(x,y)"] \arrow[rr, bend right, "\mathbb{A}(x,z)"'] & {} & {|z|} \end{tikzcd}$$
in $\mathcal{B}$.

satisfying the unit and associativity axioms.

For any bicategory $\mathcal{B}$ and any $\mathcal{B}$-category $\mathbb{A}$, there exists a bicategory $\mathcal{B} \downarrow \mathbb{A}$ with

$$(\mathcal{B}\text{-}\mathbf{Cat})/\mathbb{A} \simeq (\mathcal{B} \downarrow \mathbb{A})\text{-}\mathbf{Cat}.$$

[Fujii and Lack, The oplax limit of an enriched category, 2024].

Monoidal cat. $(\mathbf{Set}, 1, \times)$ = 1-obj. bicategory $\Sigma\mathbf{Set}$.

In particular, for any $(\mathbf{Set}\text{-})$ category $\mathcal{C}$, $\exists$ bicategory $\Sigma\mathbf{Set} \downarrow \mathcal{C}$ with

$$\mathbf{Cat}/\mathcal{C} \simeq (\Sigma\mathbf{Set} \downarrow \mathcal{C})\text{-}\mathbf{Cat}.$$

$\Sigma\mathbf{Set} \downarrow \mathcal{C}$: bicategory s.t.

- $\mathrm{ob}(\Sigma\mathbf{Set} \downarrow \mathcal{C}) = \mathrm{ob}(\mathcal{C})$ ← $\Sigma\mathbf{Set} \downarrow \mathcal{C}$: monoidal cat only when $\mathcal{C}$ = monoid M. ($\Rightarrow \Sigma\mathbf{Set} \downarrow \mathcal{C} = \mathbf{Set}/M$ with the canonical monoidal str.)
- $\forall A, B \in \mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\Sigma\mathbf{Set} \downarrow \mathcal{C})$,
$$(\Sigma\mathbf{Set} \downarrow \mathcal{C})(A, B) = \mathbf{Set}/\mathcal{C}(A,B).$$

So a $(\Sigma\mathbf{Set} \downarrow \mathcal{C})$-category $\mathbb{E}$ consists of

- a set $\mathrm{ob}(\mathbb{E})$ t.w. a function $\mathrm{ob}(\mathbb{E}) \xrightarrow{|-|} \mathrm{ob}(\Sigma\mathbf{Set} \downarrow \mathcal{C}) = \mathrm{ob}(\mathcal{C})$, $X \longmapsto UX$
- $\forall X, Y \in \mathrm{ob}(\mathbb{E})$, an object $\mathbb{E}(X,Y) = \begin{pmatrix} \mathcal{E}(X,Y) \\ \downarrow U_{X,Y} \\ \mathcal{C}(UX,UY) \end{pmatrix} \in \mathbf{Set}/\mathcal{C}(UX,UY)$

$\vdots$

$\Longrightarrow$ A category $\mathcal{E}$ equipped with a functor $\begin{matrix} \mathcal{E} \\ \downarrow U \\ \mathcal{C} \end{matrix}$.

Recall

$\boxed{\text{Q2.}}$ Is $\begin{array}{c} \text{Eilenberg-Moore} \\ \boxed{\begin{tikzcd} \mathcal{C}^T \arrow[d, "U^T"] \\ \mathcal{C} \end{tikzcd}} \\ \mathbb{C}^T \end{array}$ a (co)completion of $\begin{array}{c} \text{Kleisli} \\ \boxed{\begin{tikzcd} \mathcal{C}_T \arrow[d, "U_T"] \\ \mathcal{C} \end{tikzcd}} \\ \mathbb{C}_T \end{array}$ in a suitable sense?

$(\Sigma\mathrm{Set} \downarrow \mathcal{C})$-categories

$\boxed{\text{A2.}}$ [Bourke and F.] $\mathbb{C}^T$ is the free completion of $\mathbb{C}_T$ under lifts of absolute colimits in $\mathcal{C}$.

$= \Phi$-colimits for a suitable class Φ of weights in $(\Sigma\mathrm{Set} \downarrow \mathcal{C})$-enriched category theory.

In concrete terms, this means:

- $\begin{tikzcd} \mathcal{C}^T \arrow[d, "U^T"] \\ \mathcal{C} \end{tikzcd}$ lifts absolute colimits in $\mathcal{C}$

  $\forall\ \mathcal{D}^{op} \xrightarrow{W} \mathrm{Set}.\ \forall\ \mathcal{D} \xrightarrow{X} \mathcal{C}^T.$
  $\forall$ weighted colimit $W * U_T X$ in $\mathcal{C}$ which is an absolute colimit,
  a weighted colimit $W * X$ exists in $\mathcal{C}^T$ and satisfies $U^T(W*X) = W * U_T X$.

- $\forall\ \begin{tikzcd} \mathcal{A} \arrow[d, "V"] \\ \mathcal{C} \end{tikzcd}$ which lifts absolute colimits in $\mathcal{C}$,

$$\Phi\text{-}\mathrm{Cocts}\left(\begin{array}{c} \mathcal{C}^T \\ \downarrow U^T \\ \mathcal{C} \end{array}, \begin{array}{c} \mathcal{A} \\ \downarrow V \\ \mathcal{C} \end{array} \right) \simeq \mathrm{Cat}/\mathcal{C}\left(\begin{array}{c} \mathcal{C}_T \\ \downarrow U_T \\ \mathcal{C} \end{array}, \begin{array}{c} \mathcal{A} \\ \downarrow V \\ \mathcal{C} \end{array} \right).$$

preserving lifts of absolute colimits in $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C}^T \arrow[rr, "H"] \arrow[dr, "U^T"'] & & \mathcal{A} \arrow[dl, "V"] \\ & \mathcal{C} & \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{C}_T \arrow[r, "K"] \arrow[dr, "U_T"'] & \mathcal{C}^T \arrow[r, "H"] \arrow[d, "U^T"] & \mathcal{A} \arrow[dl, "V"] \\ & \mathcal{C} & \end{tikzcd}$$

Cor. (Monadicity theorem)

$$\begin{array}{c}\mathcal{A}\\ \big\downarrow V\\ \mathcal{C}\end{array} : \text{functor}$$

Suppose

- V has a left adjoint
- $(\mathcal{A}, V)$: lifts absolute colimits in $\mathcal{C}$

  Φ-cocomplete
- V is conservative.

Then $\mathcal{A} \simeq \mathcal{C}^T$ where T is the monad on $\mathcal{C}$ induced by V and its left adjoint.

Proof

One can show that under these assumptions

$$\begin{tikzcd} \mathcal{C}_T \arrow[rr, "J"] \arrow[dr, "U_T"'] & & \mathcal{A} \arrow[dl, "V"] \\ & \mathcal{C} & \end{tikzcd}$$

is a free completion under Φ-colimits.

Hence the claim follows from the uniqueness up to equivalence of free completions. $\square$

We can also describe the free completion under Φ-colimits.

Given $\begin{tikzcd} \mathcal{A} \arrow[d, "V"] \\ \mathcal{C} \end{tikzcd}$, its free completion $\begin{tikzcd} \Phi(\mathcal{A}) \arrow[d, "\Phi(V)"] \\ \mathcal{C} \end{tikzcd}$ under Φ-colimits is given as follows. [Betti, Cocompleteness over coverings, 1985]

<u>obj.</u> $(\mathcal{A}^{op} \xrightarrow{W} \mathbf{Set},\ C \in \mathcal{C},\ W \overset{\alpha}{\Longrightarrow} \mathcal{C}(V-, C))$ s.t.
α exhibits C as the weighted colimit $W * V$ in $\mathcal{C}$, and moreover $W * V$ is an absolute colimit.

"1-step completion" works since Φ is a saturated class of weights.

Prop. [Bourke and F.]

$\begin{tikzcd} \mathcal{A} \arrow[d, "V"] \\ \mathcal{C} \end{tikzcd}$ Suppose V has a left adjoint G. Then in

$$\begin{tikzcd} \mathcal{A}^{op} \arrow[r, "W"] \arrow[d, "V^{op}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathbf{Set} \arrow[d, "id"] \\ \mathcal{C}^{op} \arrow[r, "\mathcal{C}(-{,}C)"'] & \mathbf{Set} \end{tikzcd} \quad \overset{\text{mate}}{\longleftrightarrow} \quad \begin{tikzcd} \mathcal{A}^{op} \arrow[r, "W"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathbf{Set} \\ \mathcal{C}^{op} \arrow[r, "\mathcal{C}(-{,}C)"'] \arrow[u, "G^{op}"] & \mathbf{Set} \arrow[u, "id"'] \end{tikzcd}$$

we have $(W, C, \alpha) \in \Phi(\mathcal{A}) \iff \beta$: iso.

Cor. [Bourke and F.]

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift left, "V"] \arrow[d, phantom, "\dashv"] \\ \mathcal{C} \arrow[u, shift left, "G"] \end{tikzcd} \Longrightarrow \begin{tikzcd} \Phi(\mathcal{A}) \arrow[r] \arrow[d, "\Phi(V)"'] \arrow[dr, phantom, "\swarrow\cong"] & {[\mathcal{A}^{op}, \mathbf{Set}]} \arrow[d, "{[G^{op}, \mathbf{Set}]}"] \\ \mathcal{C} \arrow[r, "Y"'] & {[\mathcal{C}^{op}, \mathbf{Set}]} \end{tikzcd}$$

: bipullback (in fact, iso-comma obj.)

← Yoneda embedding

Cor. (Linton's theorem)

$\mathcal{C}$: category
T : monad on $\mathcal{C}$

$$\Longrightarrow \begin{tikzcd} \mathcal{C}^T \arrow[r] \arrow[d, "U^T"'] & {[\mathcal{C}_T^{op}, \mathbf{Set}]} \arrow[d, "{[F_T^{op}, \mathbf{Set}]}"] \\ \mathcal{C} \arrow[r, "Y"'] & {[\mathcal{C}^{op}, \mathbf{Set}]} \end{tikzcd}$$

: bipullback (in fact, pullback)