

Monad theory through enrichment

Soichiro Fujii
Masaryk University

(Joint work with John Bourke)

\mathcal{C} : category

$T: \mathcal{C} \rightarrow \mathcal{C}$: monad

$$\Rightarrow \begin{array}{ccc} \mathcal{C}_T & \xrightarrow{\quad K \quad} & \mathcal{C}^T \\ \downarrow U_T & & \downarrow U^T \\ \mathcal{C} & & \end{array}$$

Kleisli Eilenberg-Moore

$\mathcal{C}_T \cong$ full subcat. of \mathcal{C}^T consisting of the free T -algebras $(TA, \frac{T^2A}{TA}, \mu_{TA})$.

Every T -algebra $(A, \frac{T^2A}{A}, \mu_A)$ is the canonical coequalizer of

free T -algebras:

$$\left(TA, \frac{T^3A}{T^2A}, \mu_{TA} \right) \xrightarrow[T\alpha]{\mu_A} \left(TA, \frac{T^2A}{TA}, \mu_A \right) \xrightarrow{\alpha} \left(A, \frac{T^2A}{A}, \mu_A \right) : \text{coeq. in } \mathcal{C}^T.$$

Q1. Is \mathcal{C}^T a "(co)completion" of \mathcal{C}_T in a suitable sense?

A1. When e.g. $\mathcal{C} = \text{Set}$, \mathcal{C}^T is the exact completion of the weakly left exact category \mathcal{C}_T . [Vitale, Left covering functors, 1994]

What about general \mathcal{C} ?

In general, \mathcal{C}^T might not be exact, might not have (reflexive) coequalizers, In fact, \mathcal{C}^T could be any category!

However, \mathcal{E}^T has certain (co)completeness relative to \mathcal{E} .

- \mathcal{E}^T lifts limits in \mathcal{E} :

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{X} & \mathcal{E}^T \\ & \downarrow U^T & \\ & \mathcal{C} & \end{array}$$

If $\lim U^T X$ exists in \mathcal{E} , then
 $\lim X$ exists and $U^T(\lim X) \cong \lim U^T X$.
 $(U^T \text{ creates limits.})$

- \mathcal{E}^T lifts colimits in \mathcal{E} preserved by T and T^2 .

- In particular, \mathcal{E}^T lifts absolute colimits in \mathcal{E} .

The coequalizer

$$\left(T^2 A, \frac{T^3 A}{\begin{matrix} \downarrow \mu_{TA} \\ TA \end{matrix}} \right) \xrightarrow[\begin{matrix} \xrightarrow{\mu_A} \\ \xrightarrow{T\alpha} \end{matrix}]{} \left(TA, \frac{T^2 A}{\begin{matrix} \downarrow \mu_A \\ TA \end{matrix}} \right) \xrightarrow{\alpha} (A, \frac{TA}{A})$$

is a lifting of an absolute (in fact, split) coequalizer in \mathcal{E} :

$$\begin{array}{ccccc} T^2 A & \xrightarrow{\mu_A} & TA & \xrightarrow{\alpha} & A \\ \xleftarrow[T\alpha]{\eta_{TA}} & & \xleftarrow{\eta_A} & & \end{array}$$

⇒ Instead of the ordinary category theory, let us work in
the category theory over \mathcal{E} .

Eilenberg-Moore

Q2. Is $\frac{\mathcal{E}^T}{\mathcal{E}}$ a (co)completion of $\frac{\mathcal{C}_T}{\mathcal{E}}$ in a suitable sense?

Kleisli

... but what exactly is "category theory over \mathcal{E} "?

We take the following view: (weighted) (co)limits, free (co)completion, ...

category theory over \mathcal{C} = enriched category theory
over a bicategory $\Sigma \text{Set} \downarrow \mathcal{C}$.

[Fujii and Lack, The oplax limit of an enriched category, 2024]

$$\text{Cat}(\mathcal{C}) \cong (\Sigma \text{Set} \downarrow \mathcal{C})\text{-Cat.}$$

↑
2-categories ↑

Bicategory-enriched category theory [Betti, Carboni, Street, Walters, ...]

\mathcal{B} : bicategory

A \mathcal{B} -category \mathbb{A} consists of:

- a set $\text{ob}(\mathbb{A})$ of objects equipped with a function

$$\text{ob}(\mathbb{A}) \xrightarrow{\text{1-1}} \text{ob}(\mathcal{B})$$

$\Downarrow_{\alpha} \quad \longrightarrow |x| : \text{the extent of } x.$

- $\forall x, y \in \text{ob}(\mathbb{A})$. a 1-cell $|x| \xrightarrow{\mathbb{A}(x,y)} |y|$ in \mathcal{B} .
 $\Downarrow^{\text{the hom 1-cell from } x \text{ to } y.}$

- $\forall x \in \text{ob}(\mathbb{A})$. a 2-cell $|x| \xrightarrow{\text{Id}_x} |x|$ in \mathcal{B} .

$$\begin{array}{c} \text{Id}_x \\ \Downarrow_{\text{Id}_x} \\ |x| \xrightarrow{\mathbb{A}(x,x)} |x| \text{ in } \mathcal{B}. \end{array}$$

$$\begin{array}{ccccc} & & & & \\ & \mathbb{A}(x,y) & \longrightarrow & |y| & \\ & \Downarrow_{\text{Id}_y} & & & \\ |x| & \xrightarrow{\mathbb{A}(x,y)} & |y| & \xrightarrow{\mathbb{A}(y,z)} & |z| \text{ in } \mathcal{B}. \\ & \Downarrow_{\text{Id}_y} & & & \\ & \mathbb{A}(x,z) & \longrightarrow & |z| & \end{array}$$

satisfying the unit and associativity axioms.

For any bicategory \mathcal{B} and any \mathcal{B} -category \mathcal{A} ,
there exists a bicategory $\mathcal{B} \downarrow \mathcal{A}$ with

$$(\mathcal{B}\text{-Cat})/(\mathcal{A}) \cong (\mathcal{B} \downarrow \mathcal{A})\text{-Cat}.$$

[Fujii and Lack, The oplax limit of an enriched category, 2024].

Monoidal cat. $(\text{Set}, \mathbb{I}, \times)$ = 1-obj. bicategory ΣSet .

In particular, for any $(\text{Set}-)$ category \mathcal{C} , \exists bicategory $\Sigma \text{Set} \downarrow \mathcal{C}$
with

$$\text{Cat}/\mathcal{C} \cong (\Sigma \text{Set} \downarrow \mathcal{C})\text{-Cat}.$$

$\Sigma \text{Set} \downarrow \mathcal{C}$: bicategory s.t.

- $\text{ob}(\Sigma \text{Set} \downarrow \mathcal{C}) = \text{ob}(\mathcal{C}) \leftarrow \Sigma \text{Set} \downarrow \mathcal{C}$: monoidal cat
only when \mathcal{C} = monoid M .

$(\Rightarrow \Sigma \text{Set} \downarrow \mathcal{C} = \text{Set}/M$ with
the canonical monoidal str.)

- $\forall A, B \in \text{ob}(\mathcal{C}) = \text{ob}(\Sigma \text{Set} \downarrow \mathcal{C})$.

$$(\Sigma \text{Set} \downarrow \mathcal{C})(A, B) = \text{Set}/\mathcal{C}(A, B).$$

So a $(\Sigma \text{Set} \downarrow \mathcal{C})$ -category \mathbb{E} consists of

- a set $\text{ob}(\mathbb{E})$ t.w. a function $\text{ob}(\mathbb{E}) \xrightarrow{\dashv \dashv} \text{ob}(\Sigma \text{Set} \downarrow \mathcal{C}) = \text{ob}(\mathcal{C})$

$$X \xleftarrow{\quad} \xrightarrow{\quad} UX$$

- $\forall x, r \in \text{ob}(\mathbb{E})$, an object $\mathbb{E}(x, r) = \begin{pmatrix} \mathcal{E}(x, r) \\ \mathcal{E}(ux, xr) \\ \mathcal{E}(ux, ur) \end{pmatrix} \in \text{Set}/\mathcal{C}(ux, ur)$

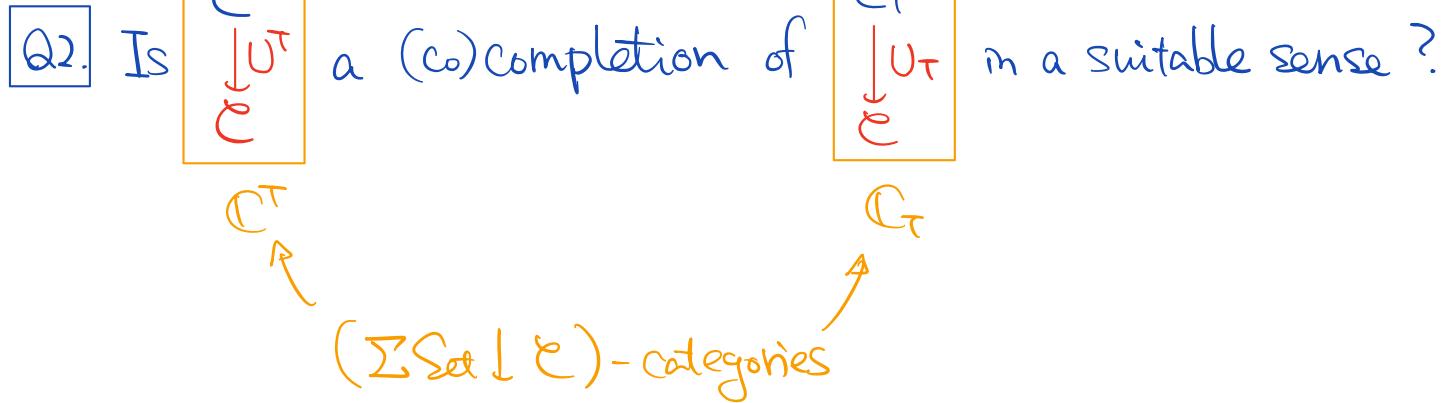
\vdots

\Rightarrow A category \mathcal{E} equipped with a functor $\begin{array}{c} \mathcal{E} \\ \downarrow U \\ \mathcal{C} \end{array}$.

Recall

Eilenberg-Moore

Kleisli



A2. [Bourke and F.] \mathcal{C}^T is the free completion of \mathcal{C} under lifts of absolute colimits in \mathcal{C} .

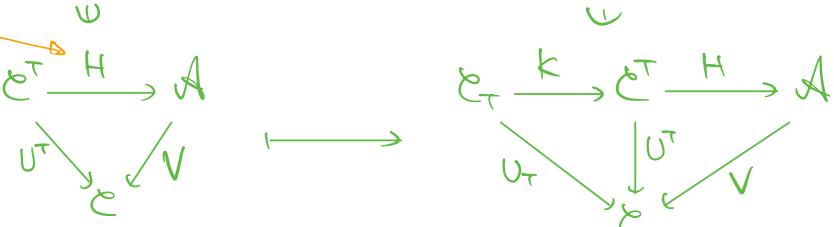
= Φ -Colimits for a suitable class Φ of weights
in $(\Sigma \text{Set} \dashv \mathcal{C})$ -enriched category theory.

In concrete terms, this means:

- $\mathcal{C}^T \downarrow U^T$ lifts absolute colimits in \mathcal{C}
 - forall $D \xrightarrow{W} \text{Set}$. $\forall D \xrightarrow{X} \mathcal{C}^T$.
 - forall weighted colimit $W * U^T X$ in \mathcal{C} which is an absolute colimit,
 - a weighted colimit $W * X$ exists in \mathcal{C}^T and satisfies $U^T(W * X) = W * U^T X$.
- $\mathcal{C} \downarrow V$ which lifts absolute colimits in \mathcal{C} ,

$$\Phi\text{-Cocts}\left(\mathcal{C}^T \downarrow U^T, \mathcal{C} \downarrow V\right) \simeq \text{Cat}/\mathcal{C}\left(\mathcal{C}^T \downarrow U^T, \mathcal{C} \downarrow V\right).$$

preserving
lifts of absolute
colimits in \mathcal{C}



Cor. (Monadicity theorem)

$\begin{array}{c} A \\ \downarrow v : \text{functor} \\ C \end{array}$

Suppose

- V has a left adjoint
- $(A, V) : \underline{\text{lifts absolute colimits in } C}$
- V is conservative. Φ -cocomplete

Then $A \simeq C^T$ where T is the monad on C induced by V and its left adjoint.

Proof |

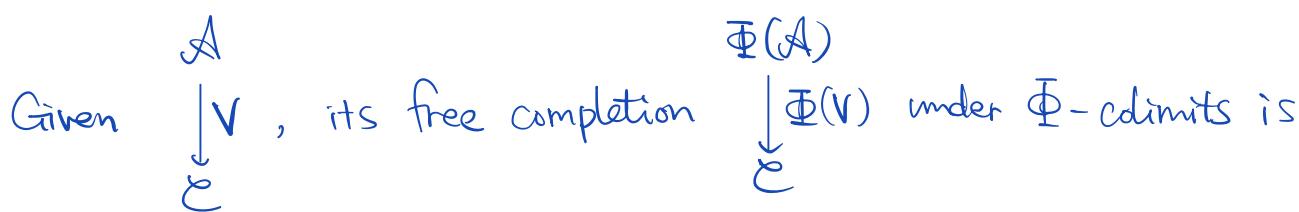
One can show that under these assumptions

$$\begin{array}{ccc}
 C_T & \xrightarrow{J} & A \\
 \downarrow U_T & & \downarrow V \\
 C & &
 \end{array}$$

is a free completion under Φ -colimits.

Hence the claim follows from the uniqueness up to equivalence of free completions. □

We can also describe the free completion under Φ -colimits.



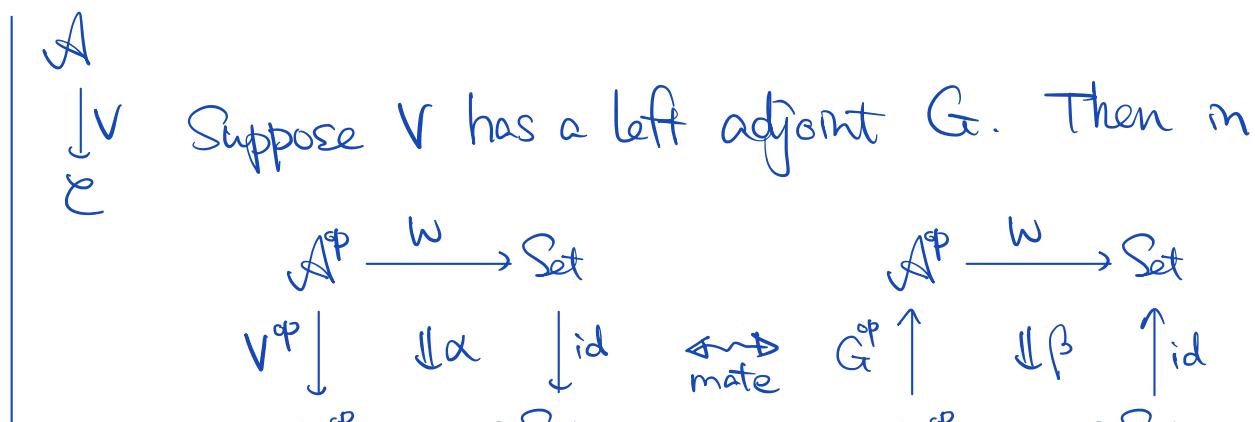
given as follows. [Betti, Cocompleteness over coverings, 1985]

obj. $(A^{\oplus} \xrightarrow{W} \text{Set}, C \in \mathcal{C}, W \xrightarrow{\alpha} \mathcal{E}(V-, C))$ s.t.

α exhibits C as the weighted colimit $W * V$ in \mathcal{C} , and moreover $W * V$ is an absolute colimit.

"1-step completion" works since Φ is a saturated class of weights.

Prop. [Bourke and F.]



we have $(W, C, \alpha) \in \Phi(A) \iff \beta : \text{iso.}$

Cor. [Bourke and F.]

$$\begin{array}{ccc} A & \Phi(A) \longrightarrow [A^{\oplus}, \text{Set}] \\ G \uparrow \dashv \downarrow V & \Rightarrow \Phi(V) \downarrow \cong [\mathcal{C}^{\oplus}, \text{Set}] & : \text{bipullback} \\ \mathcal{C} & \xrightarrow{\gamma} [\mathcal{C}^{\oplus}, \text{Set}] & (\text{in fact, iso-comma obj.}) \\ & & \mathcal{E} \text{ Tondeda embedding} \end{array}$$

Cor. (Linton's theorem)

$$\begin{array}{ccc} \mathcal{C} : \text{category} & \mathcal{E}^T \longrightarrow [\mathcal{C}_T^{\oplus}, \text{Set}] \\ T : \text{monad on } \mathcal{C} & \Rightarrow U^T \downarrow \cong [\mathcal{F}_T^{\oplus}, \text{Set}] & : \text{bipullback} \\ & & \xrightarrow{\gamma} [\mathcal{C}^{\oplus}, \text{Set}] & (\text{in fact, pullback}) \end{array}$$