Multicategorical Meta-Theorem and Completeness of Restricted Algebraic Deduction Systems

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PSSL109
Leiden University

Outline

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- $$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, n \in \mathbb{N},$$

  from commutative semirings in sets to commutative semirings in symmetric diagonal multicategories.

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  - $xyzRxyxx \leftrightsquigarrow (1, 2, 1, 1) : [4] \to [3]$.

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There are exactly 8 different modelable context structures!

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$$\theta'_{k_1,\dots,k_n}\colon [L_m] \to [K_n], L_{i-1} + x \mapsto K_{\theta(i)-1} + x, \text{ for } x \in [k_{\theta(i)}]$$

$$L_i = k_{\theta(1)} + \cdots + k_{\theta(i)} \text{ and } K_j = k_1 + \cdots + k_j$$

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$\Delta_J \subset \Delta \subset \Delta^J$ for any structure category Δ, J the set of cardinalities of the fibers of functions in Δ.

Examples of Structure Categories

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Structure monoids $I_N = \{1, n \mid n \geq N\}$ for $N \in \mathbb{N}$ induce infinitely many structure categories Δ^I.

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$$x + y \approx_{xy} y + x$$
$$f(x + y) \approx_{xyz} (f(x) + f(y)) + z$$

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$$E \vdash_R t_1 \approx_{v_1 \cdots v_n} t_2 \Rightarrow E \vdash_R s_1(t_1) \approx_w s_2(t_2)$$

for type-preserving $s_1, s_2 : \mathrm{Var}(v) \to Term$, where $wRw_1 \cdots w_n$ and $E \vdash_R s_1(v_i) \approx_{w_i} s_2(v_i)$ for $i \leq n$.

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- A composition operation
  $\circ : C(b_1 \cdots b_n, c) \times C(a^1, b_1) \times \cdots \times C(a^n, b_n) \to C(a^1 \cdots a^n, c)$.

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Let Δ be a structure category. Then C is called a **Δ-multicategory** if, for each morphism $\theta\colon [m] \to [n]$ in Δ, there exists an action map:

$$\theta^*_{a_1 \cdots a_n, b}\colon C(a_{\theta(1)} \cdots a_{\theta(m)}, b) \to C(a_1 \cdots a_n, b),$$

which respects the Δ-structure and satisfies the Δ-multicategory action axioms.

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- **Composition with Action 1:**

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- **Composition with Action 2:**
$$\tau^*_{b,c}(g) \circ (f_1, \ldots, f_n) = (\tau'_{k_1,\ldots,k_n})^*_{a,c}(g \circ (f_{\tau(1)}, \ldots, f_{\tau(m)})),$$

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Definition (σ-model)

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$$m_v(t) = \begin{cases} m^*_{v,()}(m(c)), & \text{if } t = c \text{ is a constant} \\ m^*_{v,v_i}(id), & \text{if } t = v_i \\ m^*_{v,w_1\cdots w_n}(m(f)(m_{w_1}(t_1), \ldots, m_{w_n}(t_n))), & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

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Soundness and Completeness Theorem: For an R-theory $E \cup \{t_1 \approx_v t_2\}$ with modelable R:

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- **Set-Completeness:** If R balanced or cartesian, $E \vDash_{\mathbf{Set}} t_1 \approx_v t_2 \Rightarrow E \vdash_R t_1 \approx_v t_2$

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- A Δ_R-multicategory provides a **semantic universe** for usual equational reasoning for modelable R.
- **Multicategorical Meta-Theorem:** Transfers properties from the cartesian multicategory of sets to Δ–multicategories for six different Δ.
- **Two-dimensional generalization**: Potential to generalize equational results (equations of 2-cells) from **Cat** to any $2, \Delta_R$-multicategory C for modelable balanced or cartesian R.

Acknowledgment and References

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David Forsman.
On the multicategorical meta-theorem and the completeness of restricted algebraic deduction systems, 2024.

Construction of the Balanced E-Model in Sets

$\sigma = (S, F)$ an R-signature with modelable and balanced R, $E \cup \{\phi\}$ an R-theory. Balanced E-model m witnesses completeness $m \vDash \phi$ if and only if $E \vdash_R \phi$:

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- $m = n/\sim$.