Category Theory Reverse-the-Arrows Duality Across the Sciences

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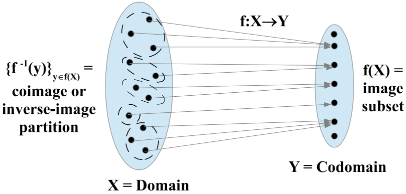
A Fundamental Duality in the Sciences

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quantitative logic	Logical probability	Logical information
CT Duality	Subobj. & limits	Quot. obj.& colimits
Physics	Classical: fully definite	Quantum: indefinite
Biology	Selectionist Mechanism	Generative Mech.

Duality starts in the dual logics of subsets & partitions: I

- Boolean logic mis-specified as logic of *propositions*; should be logic of *subsets*.
- Category theory duality gives subset-partition duality:



- *"The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition."* (Lawvere) Also Partition = Equivalence relation = Quotient set.

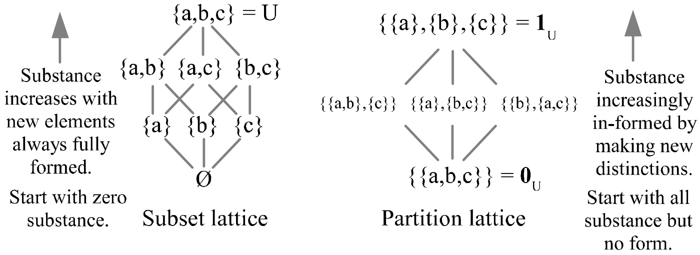
Logical algebras of subsets and partitions: I

- Given universe set $U = \{u_1, ..., u_n\}$, there is the *Boolean algebra of subsets* $\wp(U)$ with inclusion as partial ordering and the usual union and intersection, and enriched with implication or conditional: $S \supset T := S^c \cup T$ for $S, T \subseteq U$.
- A *partition* $\pi = \{B_1, ..., B_m\}$ on U is a set of non-empty subsets (blocks) of U that are disjoint and union is U.
- A *distinction* or *dit* of π is an ordered pair of elements of U in different blocks of π. The set of all dits of π is $\operatorname{dit}(\pi)$ and its complement in $U \times U - \operatorname{dit}(\pi) = \operatorname{indit}(\pi)$ is the associated equivalence class.
- Given universe set U, there is the *algebra of partitions* $\Pi(U)$ with join and meet enriched by implication where refinement is the partial ordering: $\sigma \precsim \pi$ defined by $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$.

Logical algebras of subsets and partitions: II

- Most refined partition is discrete partition $\mathbf{1}_U = \{\{u_1\}, \{u_2\}, ..., \{u_n\}\}$.

$$S \subseteq T \text{ iff } S \supset T = U.$$
$$\sigma \preceq \pi \text{ iff } \sigma \Rightarrow \pi = \mathbf{1}_U$$



Logical algebras of subsets and partitions: III

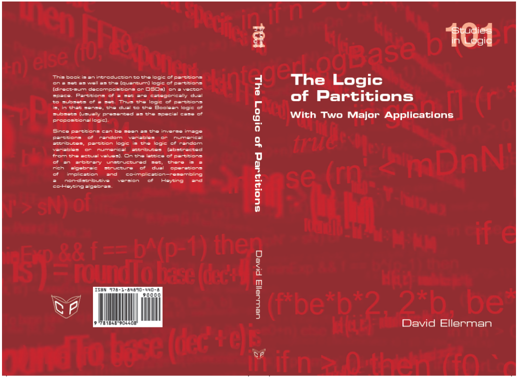
- Elements (Its) of subsets are dual to distinctions (Dits) of partitions.

Lattice of subsets $\wp(U)$	Lattice of partitions $\Pi(U)$
Its = Elements of subsets	Dits = Distinctions of partitions
1 is separator	2 is coseparator
PO Incl. of subsets $S \subseteq T$	PO $\sigma \precsim \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Join: $S \vee T = S \cup T$	Join: $\operatorname{dit}(\sigma \vee \pi) = \operatorname{dit}(\sigma) \cup \operatorname{dit}(\pi)$
Top: U all elements	Top: $\mathbf{1}_U$ with all dits
Bottom: $\emptyset$ no elements	Bottom: $\mathbf{0}_U$ with no dits

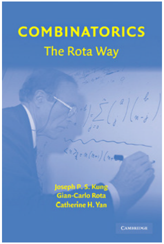
Tautologies in subset and partition logics: I

- A *subset tautology* is any formula which evaluates to U ($|U| \geq 1$) regardless of which subsets were assigned to the atomic variables.
- A *partition tautology* is any formula which always evaluates to $\mathbf{1}_U$ (the discrete partition) regardless of which partitions on U ($|U| \geq 2$) were assigned to the atomic variables.
- Every partition tautology is a subset tautology since $\Pi(2) \cong \wp(1)$.
- Partition tautologies neither included in nor include Intuitionistic tautologies (Heyting algebra validities). The weak law of excluded middle $\neg\sigma \vee \neg\neg\sigma$ is a partition but not an intuitionistic tautology and distributivity is a intuitionistic but not partition tautology.

Tautologies in subset and partition logics: II



Quant. Partition Logic = *Logical* Entropy: I



"The lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability."

$$\frac{\text{Subsets}}{\text{Probability}} \approx \frac{\text{Partitions}}{\text{Information}}$$

- Boole's quantitative logic of subsets = finite prob. theory = normalized number of elements in $S \subseteq U$, i.e., $\Pr(S) = \frac{|S|}{|U|}$.

Quant. Partition Logic = *Logical* Entropy: II

- Rota, in his Fubini Lectures, said since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure "which will capture some property that will turn out to be for [partitions] what size is to a set."
- Answer is *distinctions* or *dits* of a partition, i.e.,ordered pairs of elements in different blocks.

$$\frac{\text{Subsets}}{\text{Probability}} \approx \frac{\text{Partitions}}{\text{Information}} \text{ and } \frac{\text{Its}}{\text{Subsets}} \approx \frac{\text{Dits}}{\text{Partitions}}$$

- Then the definition of "logical information" or *logical entropy* is clear:

Quant. Partition Logic = *Logical* Entropy: III

$$h(\pi) = \frac{|\text{dit}(\pi)|}{|U\times U|} = \frac{|U\times U| - \left|\cup_j B_j \times B_j\right|}{|U\times U|} = 1 - \sum_j \left(\frac{|B_j|}{|U|}\right)^2 =$$
$$1 - \sum_j \Pr(B_j)^2 = \sum_{j\neq k} \Pr(B_j)\Pr(B_k).$$

- For a probability dist. on U, $p = (p_1, ..., p_n)$, $h(p) = 1 - \sum_i p_i^2 = \sum_{j\neq k} p_j p_k$.
- $\Pr(S)$ = prob. of one draw from U getting an it of S.
- $h(\pi)$ = prob. of two draws from U getting a dit of π.
- $h(p)$ = prob. of two draws of different indices p_i and p_j.
- If $p_i = 1$, its occurance gives no information, so information is measured by the complement. But there are two complements of 1.
- The additive 1-complement of p_i is $1 - p_i$.

Quant. Partition Logic = *Logical* Entropy: IV

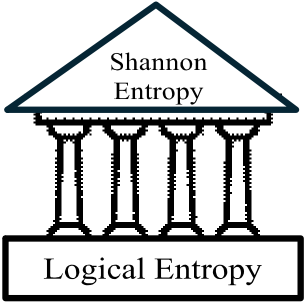
- The multiplicative 1-complement of p_i is $\frac{1}{p_i}$.
  - The additive probability average of the additive 1-complements is the logical entropy $h(p) = \sum_i p_i (1 - p_i)$.
  - The multiplicative probability average of the mult. 1-complements is the log-free Shannon entropy $\prod_i \left(\frac{1}{p_i}\right)^{p_i}$.
  - Choose a log, e.g., to base 2, and the $\log_2$ gives the usual Shannon entropy:

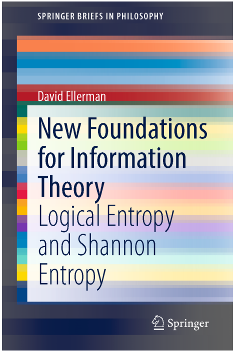
$$H(p) = \log_2 \left(\prod_i \left(\frac{1}{p_i}\right)^{p_i} \right) = \sum_i p_i \log_2 \left(\frac{1}{p_i}\right).$$

- Hence the Shannon entropy is the log of the multiplicative version of (additive) logical entropy.

Quant. Partition Logic = *Logical* Entropy: V

- Logical information theory as the quantitative version of the logic of partitions provides a new *logical* foundation for information theory.





CT Duality = interchange Its & Dits in *Sets*: I

- A binary relation $R \subseteq X \times Y$ *preserves (or transmits) elements (Its)* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.
- A binary relation $R \subseteq X \times Y$ *reflects elements (Its)* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.
- A binary relation $R \subseteq X \times Y$ *preserves (or transmits) distinctions (Dits)* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.
- A binary relation $R \subseteq X \times Y$ *reflects distinctions (Dits)* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.
- A set function $f : X \to Y$ is usually characterized as being defined everywhere and single-valued, but:

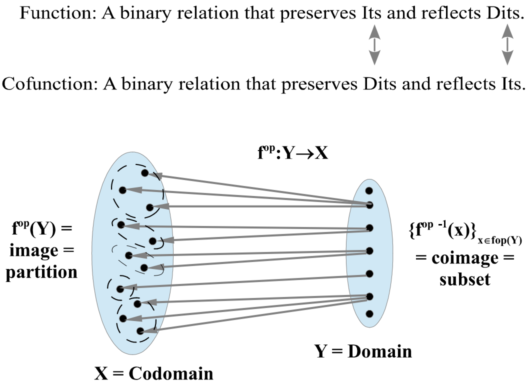
CT Duality = interchange Its & Dits in *Sets*: II

- "Defined everywhere" is the same as "preserves elements" and
- "Being single-valued" is the same as "reflecting distinctions."

Binary relation is a *function* iff it preserves Its and reflects Dits.

- Defining a function $f \subseteq X \times Y$ as a relation "everywhere defined and single-valued" gives *no hint of duality*.
- Its & Dits-definition says just interchange Its & Dits like interchanging points and lines in plane proj. geometry.
- Interchange roles of Its & Dits in a function $f \subseteq X \times Y$ gives a *cofunction* $f^{op} \subseteq Y \times X$ in $Sets^{op}$:

CT Duality = interchange Its & Dits in *Sets*: III



- *Sets* — *Sets*^{op} duality is then abstracted to make the reverse-the-arrows duality in abstract category theory.

Classical vs. Quantum Physics: Definiteness vs. Indefiniteness: I

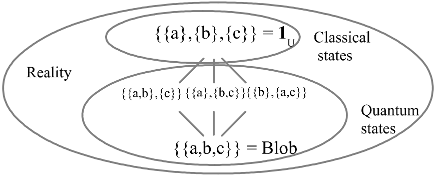
- Leibniz's Principle of Identity of Indistinguishables (PII) or Kant's Principle of Complete Determination:

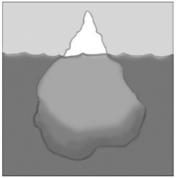
"Definite all the way down": If two entities are distinct, then there is always a distinguishing property. Hence if indistinguishable, then they are same thing.

- At logical level, math for indefiniteness = equivalence relations (i.e., partitions).
- Reduce Hilbert space vectors to sets by taking *support sets*: From particle state $|\psi\rangle = \alpha_i |u_i\rangle + \alpha_j |u_j\rangle + \alpha_k |u_k\rangle$ to partition block $\{u_i, u_j, u_k\}$.

Classical vs. Quantum Physics: Definiteness vs. Indefiniteness: II

- Partition version of PII defining classicality: For all $u, u' \in U$, if $(u, u') \in \text{indit}(\mathbf{1}_U)$, then $u = u'$.

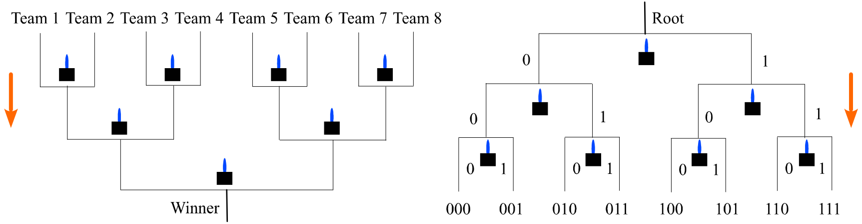




- Quantum underworld = States with indefinite values.
- Ellerman, David. 2024. "A Fundamental Duality in the Exact Sciences: The Application to Quantum Mechanics." *Foundations* 4 (2): 175–204. https://doi.org/10.3390/foundations4020013.

The Fundamental Duality in Biology

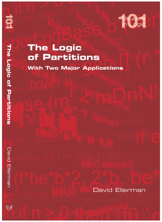
- *Generative mechanism* = a mechanism that implements codes (e.g., as distinctions or symmetry breakings), e.g., generative grammar, DNA-RNA mechanism, and stem cells.
- Dual is *Selectionist mechanism*, e.g., a single-elimination or knock-out tournament.

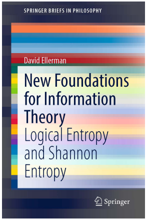


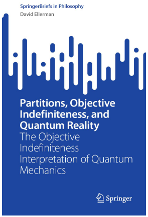
Single Elimination Tournament as a Selectionist Mechanism

Code Implementation as a Generative Mechanism

Summary







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More at: www.ellerman.org