# Towards Formal, Enriched and Homotopical Coalgebra

109th Peripatetic Seminar on Sheaves and Logic

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# Outline

# Introduction and Motivation

# Motivation

## Homotopy theory and algebraic topology for behaviour are commonplace

- Concurrent computing — detecting deadlocks[1]
- Distributed computing — computability results[2]
- Hybrid computing — detecting and handling Zeno behaviour[3]
- Smooth approximations of hybrid systems[4]
- Optimisation processes with homotopy on evolution or parameters[5]

[1] Lisbeth Fajstrup et al. *Directed Algebraic Topology and Concurrency*. Springer, 2016, p. 167. 1 p. ISBN: ISBN 978-3-319-15397-1. DOI: 10.1007/978-3-319-15398-8.

[2] Maurice Herlihy, Dmitry Kozlov, and Sergio Rajsbaum. *Distributed Computing Through Combinatorial Topology*. 1st ed. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., Nov. 2013. 336 pp. ISBN: 978-0-12-404578-1.

[3] Aaron D. Ames and S. Shankar Sastry. "Characterization of Zeno Behavior in Hybrid Systems Using Homological Methods". In: *Proc. of the 2005 American Control Conference*. ACC 2005. 2005, 1160–1165 vol. 2. DOI: 10.1109/ACC.2005.1470118.

[4] Tyler Westenbroek et al. "Smooth Approximations for Hybrid Optimal Control Problems with Application to Robotic Walking". In: *IFAC-PapersOnLine*. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021 54.5 (Jan. 1, 2021), pp. 181–186. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2021.08.495.

[5] Andreas Klingler and Tim Netzer. *Homotopy Methods for Convex Optimization*. Mar. 4, 2024. DOI: 10.48550/arXiv.2403.02095. arXiv: 2403.02095 [cs, math]. Pre-published; Xi Lin et al. *Continuation Path Learning for Homotopy Optimization*. July 24, 2023. DOI: 10.48550/arXiv.2307.12551. arXiv: 2307.12551 [cs]. Pre-published; Layne T. Watson. "Globally Convergent Homotopy Methods". In: *Encyclopedia of Optimization*. Ed. by Christodoulos A. Floudas and Panos M. Pardalos. Boston, MA: Springer US, 2009, pp. 1272–1277. ISBN: 978-0-387-74759-0. DOI: 10.1007/978-0-387-74759-0_222.

# Enriched Coalgebra

## Coalgebra and coalgebraic modal logic with extra structure

- Order enrichment[6]
- Quantitative behaviour via quantaloids[7]
- Topology and order in coalgebraic modal logic[8]
- **CPO**-enriched Kleisli categories for denotational semantics

[6] Adriana Balan, Alexander Kurz, and Jiří Velebil. "Positive Fragments of Coalgebraic Logics". In: *Logical Methods in Computer Science* Volume 11, Issue 3 (Sept. 22, 2015). ISSN: 1860-5974. DOI: 10.2168/LMCS-11(3:18)2015.

[7] Adriana Balan, Alexander Kurz, and Jiří Velebil. "Extending Set Functors to Generalised Metric Spaces". In: *Logical Methods in Computer Science* Volume 15, Issue 1 (Jan. 29, 2019). ISSN: 1860-5974. DOI: 10.23638/LMCS-15(1:5)2019.

[8] Nick Bezhanishvili, Jim de Groot, and Yde Venema. "Coalgebraic Geometric Logic: Basic Theory". In: *Logical Methods in Computer Science* Volume 18, Issue 4 (2022). DOI: 10.46298/LMCS-18(4:10)2022.

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## Homotopy theory via enrichment

- Simplicial or topological (model) categories for homotopy theory
- (Weak) Homotopy equivalence of systems
- Homotopy-invariant logic
- Homological algebra to find behavioural obstructions

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# Behavioural Obstructions in Topological Models

# Homotopy Theory via Topological Enrichment

## Topological Enrichment

Suppose $\underline{\mathcal{C}}$ is a $\mathbf{CG}$-enriched category (compactly generated spaces):

- it has a space $\underline{\mathcal{C}}(X, Y) \in \mathbf{CG}$ for all objects $X, Y$
- there are continuous composition maps $c_{X,Y,Z} : \underline{\mathcal{C}}(Y, Z) \times \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, Z)$
- there is an identity $\mathrm{id}_X : * \to \underline{\mathcal{C}}(X, X)$ for all objects $X$
- an associativity and two unit diagrams commute

Enrichment (plus other things) enables homotopy theory[9]

- Define a homotopy $h : f \Rightarrow g$ between $f, g \in \underline{\mathcal{C}}(X, Y)$ to be a continuous map $h : [0, 1] \to \underline{\mathcal{C}}(X, Y)$ with $h(0) = f$ and $h(1) = g$
- Write $f \sim g$ if there is some homotopy $f \Rightarrow g$
- Homotopy coherent nerve yields quasi-category

[9] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press, 2014. ISBN: 978-1-107-04845-4. URL: https://math.jhu.edu/~eriehl/cathtpy/; Michael Shulman. *Homotopy Limits and Colimits and Enriched Homotopy Theory*. June 30, 2009. DOI: 10.48550/arXiv.math/0610194. arXiv: math/0610194. Pre-published.

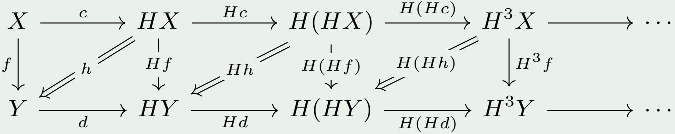
# Behaviour up to Homotopy

## Example

- $\underline{\mathbf{CG}}$ is $\mathbf{CG}$ enriched: Continuous maps form a space $\underline{\mathbf{CG}}(X, Y)$ and composition is continuous
- Functor of trajectories with duration for hybrid systems:

$$H(X) = \{(\varrho, d) \in X^{\mathbb{R}_{\geq 0}} \times [0, \infty] \mid \varrho \circ \min(-, d) = \varrho\}$$

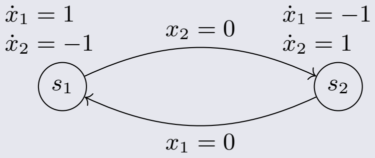
- A $\mathbf{CG}$-functor: $H_{X,Y}(f)(\varrho, d) = (f \circ \varrho, d)$ is continuous map $H_{X,Y} \colon \underline{\mathbf{CG}}(X, Y) \to \underline{\mathbf{CG}}(HX, HY)$
- Hence, homotopy $h : f \to g$ can be mapped to a homotopy $Hh \colon Hf \to Hg$ by $Hh = H_{X,Y} \circ h$
- A homotopical coalgebra morphism $(f, h) \colon c \to d$ with a homotopy $h \colon Hf \circ c \Rightarrow d \circ f$ yields

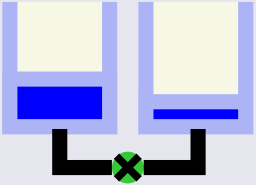


# Zeno Behaviour

## Sisyphus pumps water

- Two water tanks connected by a pump
- Pumps water until tank is empty and then switches direction
- Two states for the pumping directions
- Guards enable transitions
- Two sets of differential equations for linear flow



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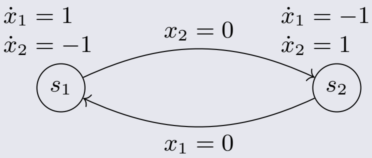


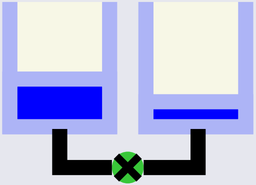

## Not physically realisable

Infinite switching in finite time when both tanks are empty

# Modelling the Water Tanks

## Domains and guards

$$\Omega_k = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_k \geq 0\}, k \in \{1, 2\}$$
$$G_1 = \{(x_1, x_2) \in \Omega_1 \mid x_2 = 0\}$$
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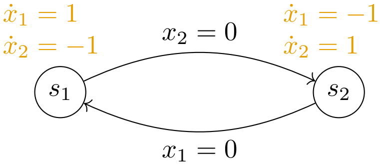

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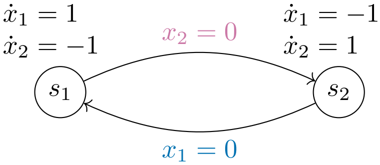

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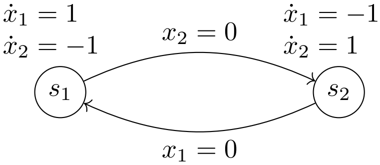
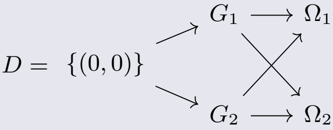
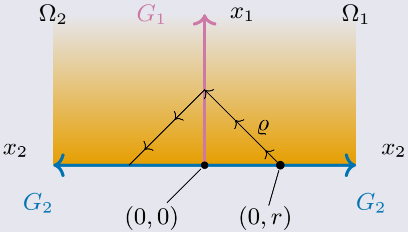



## Hybrid computation as coalgebra on colimit space




$$S_1 = \operatorname{colim} D$$
$$c_1 : S_1 \to HS_1$$



# Realisable Sisyphus

## Switching

- Switching takes time
- But it is irrelevant how much
- Trajectories in homotopy colimit $\mathrm{hcolim}\, D$ of $D$!

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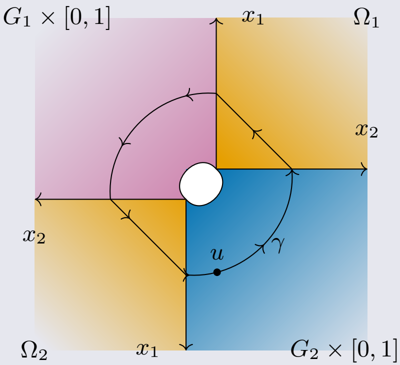

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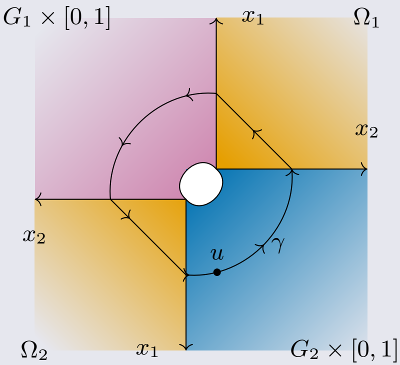


## Postulate

Any physically realisable model must have a coalgebra map up to homotopy into $c_2$.

# Homotopical Obstruction to Realisability

## Water tank pump not realisable

- Let $f\colon S_1 \to S_2$ be a map with a homotopy $h\colon c_2 \circ f \Rightarrow Hf \circ c_1$ (endpoint-preserving)

[10] Tyler Westenbroek et al. "Smooth Approximations for Hybrid Optimal Control Problems with Application to Robotic Walking". In: *IFAC-PapersOnLine*. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021 54.5 (Jan. 1, 2021), pp. 181–186. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2021.08.495.

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## Dual use

The other way around: $c_2$ forces system to be realisable[10]

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# Formal Coalgebra

# Formal Coalgebra in $2$-Categories

- Generalise category of coalgebras for endofunctors
- Coalgebra objects (special $2$-limits, inserters[11])
- Exposes distributive laws as the main tool in coalgebra

[11] Claudio Hermida and Bart Jacobs. "Structural Induction and Coinduction in a Fibrational Setting". In: *Information and Computation* 145 (1997), pp. 107–152. DOI: 10.1006/inco.1998.2725; Adriana Balan, Alexander Kurz, and Jiří Velebil. "Extending Set Functors to Generalised Metric Spaces". In: *Logical Methods in Computer Science* Volume 15, Issue 1 (Jan. 29, 2019). ISSN: 1860-5974. DOI: 10.23638/LMCS-15(1:5)2019.

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## Solutions

- Yoneda structures, proarrow equipments
- Fibrant double categories

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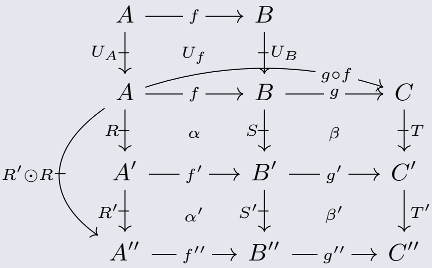
## Composition in a double category $\mathbb{C}$




$\beta \circ \alpha : \begin{bmatrix} & gf & \\ R & & T \\ & g'f' & \end{bmatrix}$

$\alpha' \odot \alpha : \begin{bmatrix} & f & \\ R' \odot R & & S' \odot S \\ & f'' & \end{bmatrix}$

$R'' \odot (R' \odot R) \cong (R'' \odot R') \odot R$

$R \odot U_A = R$ (makes life easier)

$\alpha \odot U_f = \alpha$

# Relevant Examples

- $\mathbb{C}\mathbf{at}$ — categories
  - category, functors, profunctors, natural transformations
  - procomposition is Day convolution
  - $U_A$ is hom-functor
- $\mathcal{V}$-$\mathbb{C}\mathbf{at}$ — enriched categories
  - category, enriched functors, distributors, enriched natural transformations
  - procomposition is enriched Day convolution
  - $U_A$ is enriched hom-functor
- A $2$-category $\mathcal{C}$ yields double category $\mathbb{M}\mathcal{C}$ with only identity promorphisms

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle U_A}\big\downarrow\!\!\!\!\!+ & \Downarrow{\scriptstyle \alpha} & +\!\!\!\!\!\big\downarrow{\scriptstyle U_B} \\
A & \xrightarrow{g} & B
\end{array}
$$

- $\mathbb{M}\mathbf{od}$ — Rings, homomorphisms, modules and module morphisms
- $\mathbb{S}\mathbf{et}$ — Sets, maps, relations, inclusions
- ...

# Formal Coalgebra in double categories

- Define double category $\mathbb{C}^{\circlearrowleft}$ of
  - endomorphisms $(A, f)$,
  - distributive laws $(h, \delta) \colon f \to g$,
  - endocells $(R, \gamma) \colon f \nrightarrow g$ and
  - distributive law morphisms $\alpha : \left[ (R,\gamma) \begin{smallmatrix} (h_1,\delta_1) \\ \\ (h_2,\delta_2) \end{smallmatrix} (S,\varrho) \right]$

$$
\begin{array}{c}
A \\
f \big\downarrow \\
A
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{f} & A & \xrightarrow{h} & B \\
U_A \big\downarrow & & \delta & & \big\downarrow U_B \\
A & \xrightarrow{h} & B & \xrightarrow{g} & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
R \big\downarrow & \gamma & \big\downarrow R \\
B & \xrightarrow[g]{} & B
\end{array}
\qquad
\begin{array}{ccc}
A_1 & \xrightarrow{h_1} & B_1 \\
R \big\downarrow & \alpha & \big\downarrow S \\
A_2 & \xrightarrow[h_2]{} & B_2
\end{array}
$$

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A
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A_1 & \xrightarrow{h_1} & B_1 \\
{\scriptstyle R}\downarrow & \alpha & \downarrow{\scriptstyle S} \\
A_2 & \xrightarrow[h_2]{} & B_2
\end{array}
$$

- forgetful double functor $U \colon \mathbb{C}^{\circlearrowleft} \to \mathbb{C}$ with $U(A, f) = A$
- inclusion double functor $I \colon \mathbb{C} \to \mathbb{C}^{\circlearrowleft}$ with $IA = (A, \mathrm{id}_A)$

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$$\begin{array}{c} A \\ {\scriptstyle f}\big\downarrow \\ A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{f} & A & \xrightarrow{h} & B \\ {\scriptstyle U_A}\big\downarrow\!\!\!\!+ & & \delta & & +\!\!\!\!\big\downarrow{\scriptstyle U_B} \\ A & \xrightarrow{h} & B & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle R}\big\downarrow\!\!\!\!+ & \gamma & +\!\!\!\!\big\downarrow{\scriptstyle R} \\ B & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{h_1} & B_1 \\ {\scriptstyle R}\big\downarrow\!\!\!\!+ & \alpha & +\!\!\!\!\big\downarrow{\scriptstyle S} \\ A_2 & \xrightarrow[h_2]{} & B_2 \end{array}$$

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**Theorem (Functorial construction of coalgebra objects)**

*If $\mathbb{C}$ has enough double limits, then $I$ has a right adjoint $\mathrm{CoAlg} \colon \mathbb{C}^{\circlearrowleft} \to \mathbb{C}$ and a $p \colon \mathrm{CoAlg} \to U$.*

# Fibrant Double Categories

Double category $\mathbb{C}$ is fibrant[14] if every niche

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & & \big\downarrow{\scriptstyle S} \\ A & \xrightarrow[g]{} & B \end{array} \quad \text{can be universally filled} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle S(f,g)}\big\downarrow & \overline{S}(f,g) & \big\downarrow{\scriptstyle S} \\ A & \xrightarrow[g]{} & B \end{array}$$

[14] Michael Shulman. "Framed Bicategories and Monoidal Fibrations". In: *Theory and Applications of Categories* 20.18 (2008), pp. 650–738. URL: http://www.tac.mta.ca/tac/volumes/20/18/20-18abs.html.

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Special case: companion

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle A(f,1)}\big\downarrow & \overline{A}(f,1) & \big\downarrow{\scriptstyle U_A} \\ A & = & A \end{array}$$

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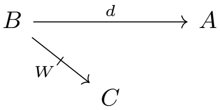
- Examples: $\mathbb{C}\mathbf{at}$, $\mathcal{V}$-$\mathbb{C}\mathbf{at}$, $\mathbb{M}\mathbf{od}$, $\mathbb{S}\mathbf{et}$
- Non-example: $\mathbb{M}\mathcal{C}$

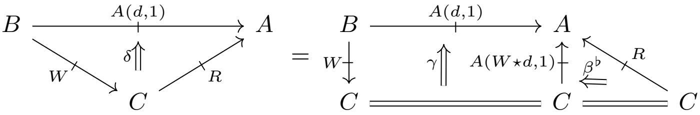
[14] Michael Shulman. "Framed Bicategories and Monoidal Fibrations". In: *Theory and Applications of Categories* 20.18 (2008), pp. 650–738. URL: http://www.tac.mta.ca/tac/volumes/20/18/20-18abs.html.

## Formal Colimits in Fibrant Double Categories

Colimit of a diagram $d$ weighted by $W$




is right Kan extension $(W \star d, \gamma)$ with $W \star d \colon C \to A$ and unique factorisation property[15]




### Theorem

*The projection* $p \colon \mathrm{CoAlg} \to U$ *creates colimits:* $W \star (p_{(A,f)} d)$ *induces* $W \star d$ *for* $d \colon B \to \mathrm{CoAlg}(A, f)$.

[15] Dominic Verity. "Enriched Categories, Internal Categories and Change of Base". PhD thesis. Cambridge University, 1992. URL: http://www.tac.mta.ca/tac/reprints/articles/20/tr20abs.html.

# Thank you for your attention!