Monotone weak distributive laws over weakly lifted powerset monads in categories of algebras[i]

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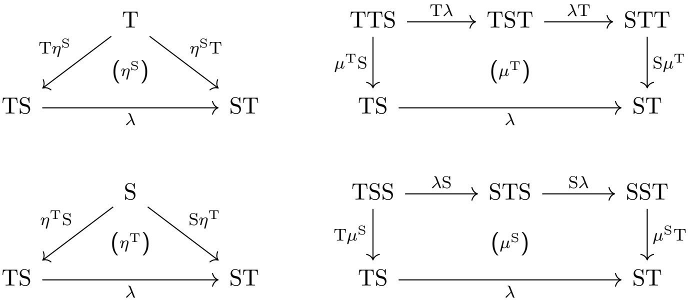
- True with a *distributive law* (Beck 1969), i.e. a $\lambda \colon \mathrm{TS} \Rightarrow \mathrm{ST}$ s.t.:



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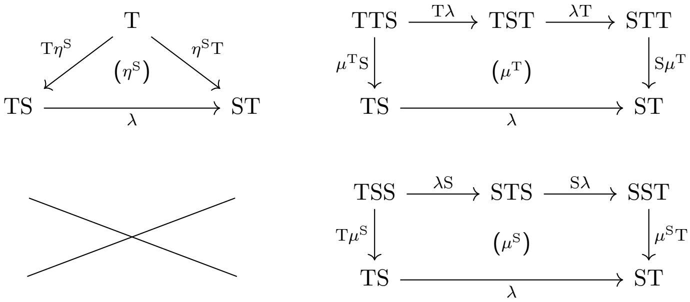

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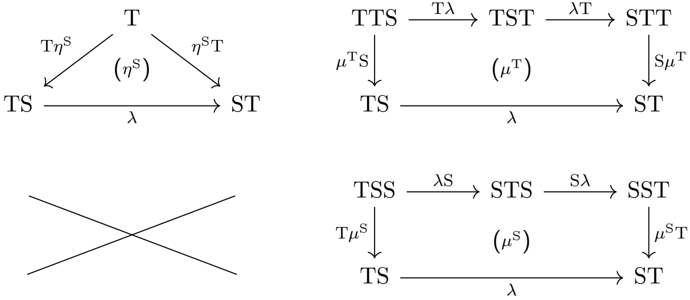

- A WDL yields a *weak composite* monad $\mathrm{S} \bullet \mathrm{T}$

Examples of weak distributive laws

- (Goy and Petrișan 2020) In Set, there is a WDL $\mathrm{DP} \Rightarrow \mathrm{PD}$ s.t. $(\mathrm{P} \bullet \mathrm{D})X = \{\text{convex subsets of prob. distr. on } X\}$

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ii

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The formula is the same... is this just a coincidence?

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Equivalent presentations of WDLs

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iv

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	$\beta\mathrm{P} \Rightarrow \mathrm{P}\beta$[iv]		$\left(\mathrm{V}, \eta^{\mathrm{V}}, \mu^{\mathrm{V}}\right)$ in $\mathrm{EM}(\beta) \cong \mathrm{KHaus}$

[iv] β is the ultrafilter monad

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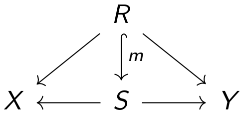

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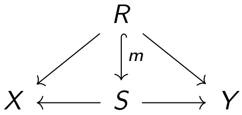

- **Examples.** Set is regular, $\mathrm{Rel}(\mathrm{Set}) = \mathrm{Rel}$. KHaus is regular, $\mathrm{Rel}(\mathrm{KHaus})$ = compact Hausdorff spaces and closed relations.

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- **Examples.** P and μ^{P} are nearly cartesian and $\mathrm{Kl}(\mathrm{P}) \cong \mathrm{Rel}$ (Garner 2020). V and μ^{V} are nearly cartesian and $\mathrm{Kl}(\mathrm{V}) \hookrightarrow \mathrm{Rel}(\mathrm{KHaus})$ (Goy, Petrișan, and Aiguier 2021).

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  - $(\mathrm{P}, \eta^{\mathrm{P}}, \mu^{\mathrm{P}})$ has a weak lifting $\left(\overline{\mathrm{P}}, \eta^{\overline{\mathrm{P}}}, \mu^{\overline{\mathrm{P}}}\right)$ to $\mathrm{EM}(\mathrm{T})$;

  Moreover:
  - $\mathrm{EM}(\mathrm{T})$ is regular;
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- **Lemma.** Weakly lifting preserves *near cartesianness*.

Weakly lifting the setting for monotone WDLs, part 1

- From now on T is a monad on Set s.t. T and μ^{T} are nearly cartesian:
  - $(\mathrm{T}, \eta^{\mathrm{T}}, \mu^{\mathrm{T}})$ has a monotone weak extension to $\mathrm{Rel} \cong \mathrm{Kl}(\mathrm{P})$;
  - there is a monotone WDL $\mathrm{TP} \Rightarrow \mathrm{PT}$;
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- **Lemma.** Weakly lifting preserves *near cartesianness*.
- **Example.** V weak lifting of P to $\mathrm{EM}(\beta) \cong \mathrm{KHaus}$: P and μ^{P} are nearly cartesian hence V and μ^{V} are as well.

Weakly lifting the setting for monotone WDLs, part 2

- Is $\mathrm{Kl}\left(\overline{\mathrm{P}}\right)$ always a subcategory of $\mathrm{Rel}(\mathrm{EM}(\mathrm{T}))$? If so, when does the relational extension of $\overline{\mathrm{S}}$ restrict to $\mathrm{Kl}\left(\overline{\mathrm{P}}\right)$?

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- **Proposition.** An $\mathrm{EM}(\mathrm{T})$-relation $(A, a) \leftrightsquigarrow (B, b)$

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    $\Rightarrow \exists t' \in \mathrm{T}R, x' = r(t') \wedge (\mathrm{T}f)(t') = t.$

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  I say f is *decomposable*.
- **Theorem.** Let $\overline{\mathrm{S}}$ be a weak lifting of a monad S with a monotone WDL $\mathrm{SP} \Rightarrow \mathrm{PS}$. There is a monotone WDL $\overline{\mathrm{S}}\overline{\mathrm{P}} \Rightarrow \overline{\mathrm{P}}\overline{\mathrm{S}}$ iff $\overline{\mathrm{S}}$ preserves decomposable morphisms.

Weakly lifting the setting for monotone WDLs to KHaus

In $\mathrm{KHaus} \cong \mathrm{EM}(\beta)$,

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  - **Corollary.** R does not preserve open maps hence there is no monotone WDL $\mathrm{RV} \Rightarrow \mathrm{VR}$.

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  - **Corollary.** R does not preserve open maps hence there is no monotone WDL $\mathrm{RV} \Rightarrow \mathrm{VR}$.
  - **Theorem.** R preserves surjective open maps hence there is a (unique) monotone WDL $\mathrm{RV}_* \Rightarrow \mathrm{V}_*\mathrm{R}$[v]. See also (Goubault-Larrecq 2024).

[v] V_*X = space of closed subsets of X, excluding $\varnothing$

Conclusion: no-go theorems for monotone WDLs

$\mathrm{PP} \Rightarrow \mathrm{PP}$ and $\mathrm{VV} \Rightarrow \mathrm{VV}$ look the same... but monotone WDLs over $\overline{\mathrm{P}}$ are quite rare otherwise:

	KHaus		JSL	Conv	Mon				CMon		
	V	R	$\overline{\mathrm{P}}$	$\overline{\mathrm{P}}$	$\overline{\mathrm{M}}$	$\overline{\mathrm{D}}$	$\overline{\mathrm{P}}$	$\overline{\mathrm{M}_S}$	$\overline{\mathrm{M}}$	$\overline{\mathrm{D}}$	$\overline{\mathrm{P}}$
$\overline{\mathrm{P}}$	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
$\overline{\mathrm{P}_*}$	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗

- What's next?
  - extending this framework: Pos-regular categories, other monads of relations
  - no-go theorems for (all) WDLs
  - seeing this in the setting of monoidal topology