Two-sided and other generalizations of weak factorization systems

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Outline

Motivation for two-sided weak factorization systems

Two-sided weak factorization systems

Generalization[1]

[1] jww Benno van den Berg, Erin McCloskey

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Motivation for two-sided weak factorization systems

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Generalization[2]

[2] jww Benno van den Berg, Erin McCloskey

Motivation

- (Type theoretic) weak factorization systems interpret the identity type of Martin-Löf type theory.

Motivation

- (Type theoretic) weak factorization systems interpret the identity type of Martin-Löf type theory.
- We want to develop a notion of *directed* notion of (type theoretic) weak factorization systems to interpret an analogous *directed* identity type.

WFS from relation

How do we get weak factorization systems from a functorial reflexive relation (Id-type) on a category?

$$X \xrightarrow{\eta} \Gamma(X) \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

WFS from relation

How do we get weak factorization systems from a functorial reflexive relation (Id-type) on a category?

$$X \xrightarrow{\eta} \Gamma(X) \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

First, we need to factor any map $f : X \to Y$. We do this using the mapping path space:

$$X \xrightarrow{\eta} X_f \times_{\epsilon_0} \Gamma(Y) \xrightarrow{\epsilon_1} Y$$

But this introduces an asymmetry.

In models of identity types, this is resolved because a 'symmetry' involution on $\Gamma(X)$ is required that preserves η and switches ϵ_0 and ϵ_1.

In the directed case (e.g. $\mathcal{C}^{\rightarrow}$), this isn't resolved and we get two factorizations underlying two weak factorization systems.

$$X \xrightarrow{\eta} X_f \times_{\epsilon_0} \Gamma(Y) \xrightarrow{\epsilon_1} Y \qquad X \xrightarrow{\eta} \Gamma(Y)_{\epsilon_1} \times_f X \xrightarrow{\epsilon_0} Y$$

We want to see these two wfs's as part of the same structure.

Relation from WFS

How do we get a functorial reflexive relation (Id-type) back from a wfs on a category?

We factor the diagonal of every object.

$$X \xrightarrow{\lambda(\Delta_X)} M(\Delta_X) \xrightarrow{\rho(\Delta_X)} X \times X$$

Relation from WFS

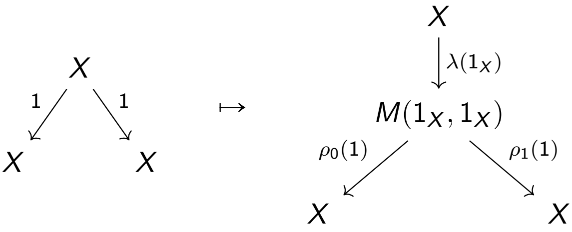
How do we get a functorial reflexive relation (Id-type) back from a wfs on a category?

We factor the diagonal of every object.

$$X \xrightarrow{\lambda(\Delta_X)} M(\Delta_X) \xrightarrow{\rho(\Delta_X)} X \times X$$

In our new notion of directed weak factorization, we need to preserve this ability.

We can think of this as the following operation.



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Generalization[3]

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Two-sided factorization

Factorization on a category

- a factorization of every morphism

$$X \xrightarrow{f} Y \qquad \mapsto \qquad X \xrightarrow{\lambda(f)} Mf \xrightarrow{\rho(f)} Y$$

- that extends to morphisms of morphisms

Two-sided factorization on a category

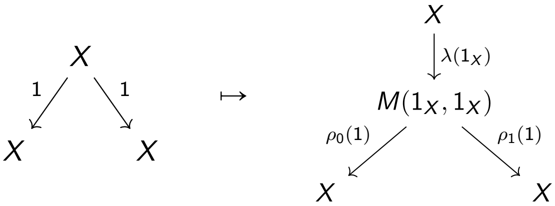
- a factorization of every span into a **sprout**

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ Y & & Z \end{array} \qquad \mapsto \qquad \begin{array}{ccc} & X & \\ & \downarrow{\scriptstyle \lambda(f,g)} & \\ & M(f,g) & \\ {\scriptstyle \rho_0(f,g)}\swarrow & & \searrow{\scriptstyle \rho_1(f,g)} \\ Y & & Z \end{array}$$

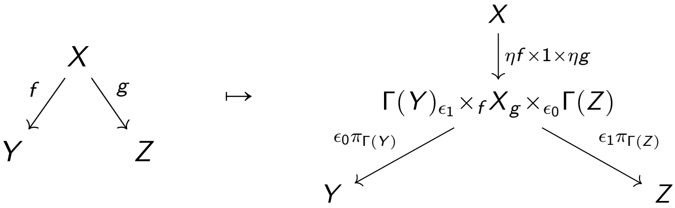
- that extends to morphisms of spans

Relations

From any two-sided factorization, we obtain a reflexive relation on every object



Conversely, from a reflexive relation $X \xrightarrow{\eta} \Gamma(X) \xrightarrow{\epsilon} X, X$ on each object, we obtain a two-sided factorization (Street 1974)



Comma category

Notation

Write a span

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ Y & & Z \end{array}$$

as $f, g : X \to Y, Z$.

Then a factorization maps

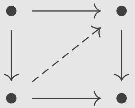
$$X \xrightarrow{f,g} Y, Z \qquad \mapsto \qquad X \xrightarrow{\lambda(f,g)} M(f, g) \xrightarrow{\rho(f,g)} Y, Z$$

We're in the comma category $\Delta_{\mathcal{C}} \downarrow \mathcal{C} \times \mathcal{C}$.

Lifting

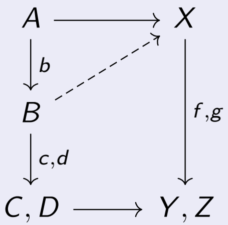
Lifting

A lifting problem is a commutative square, and a solution is a diagonal morphism making both triangles commute.



Two-sided lifting

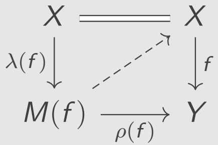
A sprout $A \xrightarrow{b} B \xrightarrow{c,d} C, D$ **lifts** against a span $X \xrightarrow{f,g} Y, Z$ if for any commutative diagram of solid arrows, there is a dashed arrow making the whole diagram commute.



Two-sided fibrations

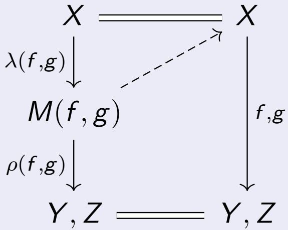
Fibrations.

Given a factorization, a **fibration** is a morphism $f : X \to Y$ for which there is a lift in



Two-sided fibrations

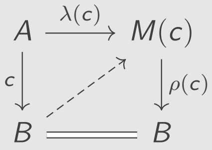
Given a two-sided factorization, a **two-sided fibration** is a span $f, g : X \to Y, Z$ for which there is a lift in



Rooted cofibrations

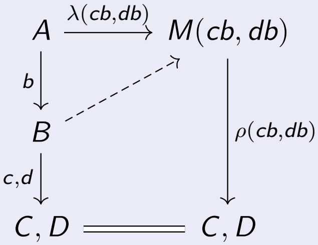
Cofibrations

Given a factorization, a **cofibration** is a morphism $c : A \to B$ for which there is a lift in



Rooted cofibrations

Given a two-sided factorization, a **rooted cofibration** is a sprout $A \xrightarrow{b} B \xrightarrow{c,d} C, D$ for which there is a lift in



First results

For a factorization...

- every isomorphism is both a cofibration and fibration
- cofibrations and fibrations are closed under retracts
- cofibrations and fibrations are closed under composition
- fibrations are stable under pullback
- cofibrations lift against fibrations

For a two-sided factorization...

- every sprout whose top morphism is an isomorphism is a rooted cofibration
- every product projection $X \times Y \to X, Y$ is a two-sided fibration
- the span-composition of two two-sided fibrations is a two-sided fibration
- two-sided fibrations are stable under pullback
- rooted cofibrations lift against two-sided fibrations

Two-sided weak factorization systems

Weak factorization system

A factorization (λ, ρ) such that $\lambda(f)$ is a cofibration and $\rho(f)$ is a fibration for each morphism f

Two-sided weak factorization system

A two-sided factorization (λ, ρ) such that the span $\rho(f, g)$ is a two-sided fibration

Strong two-sided weak factorization system

A two-sided weak factorization system such that the sprout in green is a cofibration for each span (f, g).

$$\begin{array}{ccccc}
X & = & X & = & X \\
\downarrow{\scriptstyle \lambda(f,!)} & & \downarrow{\scriptstyle \lambda(f,g)} & & \downarrow{\scriptstyle \lambda(!,g)} \\
M(f,!) & \xleftarrow{M(1,1,!)} & M(f,g) & \xrightarrow{M(1,!,1)} & M(!,g) \\
\downarrow{\scriptstyle \rho(f,!)} & & \downarrow{\scriptstyle \rho(f,g)} & & \downarrow{\scriptstyle \rho(!,g)} \\
Y, * & \xleftarrow{1,!} & Y, Z & \xrightarrow{!,1} & *, Z
\end{array}$$

Two-sided weak factorization systems

Theorem (Rosický-Tholen 2002)

In a weak factorization system, the cofibrations are exactly the morphisms with the left lifting property against the fibrations and vice versa.

Theorem

In a two-sided weak factorization system, the rooted cofibrations are exactly the morphisms with the left lifting property against the two-sided cofibrations and vice versa.

Two weak factorization systems

Proposition

Consider a 2swfs $(\lambda, \rho_0, \rho_1)$ on a category with a terminal object. This produces two weak factorization systems: a **future** wfs whose underlying factorization is given by

$$X \xrightarrow{f} Y \qquad \mapsto \qquad X \xrightarrow{\lambda(!,f)} M(!,f) \xrightarrow{\rho_1(!,f)} Y$$

and a **past** wfs whose underlying factorization is given by

$$X \xrightarrow{f} Y \qquad \mapsto \qquad X \xrightarrow{\lambda(f,!)} M(f,!) \xrightarrow{\rho_0(f,!)} Y$$

Two weak factorization systems

Proposition

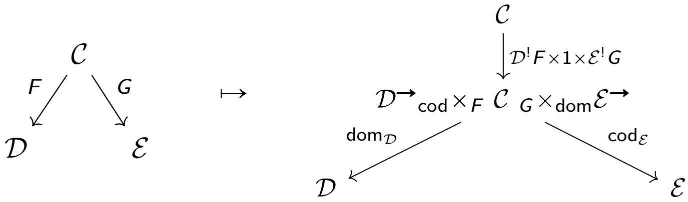
Consider a two-sided fibration $f, g : X \to Y, Z$ in a 2swfs. Then f is a past fibration and g is a future fibration.

Proposition

Consider a two-sided fibration $f, g : X \to Y, Z$ in a 2swfs, a past fibration $f' : Y \to Y'$ and $h' : Z \to Z'$. Then $f'f, g'g : X \to Y', Z'$ is a two-sided fibration.

The example in *Cat*

There is a 2swfs in *Cat* given by the factorization



- The past fibrations contain the Grothendieck fibrations
- The future fibrations contain the Grothendieck opfibrations
- The two-sided fibrations contain the (Grothendieck) two-sided fibrations (Street 1974)

Type-theoretic 2SWFSs

Theorem (N)

The following are equivalent for a wfs:

- it is generated by a weakly left transitive, weakly left connected, and weakly symmetric functorial reflexive relation $X \to \Gamma(X) \to X, X$.
- it is type-theoretic: (1) all objects are fibrant and (2) the Frobenius condition, that cofibrations are stable under pullback along fibrations, holds

Theorem

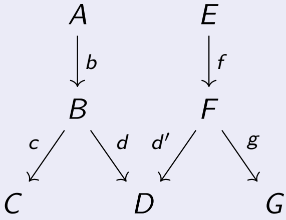
The following are equivalent for a strong 2swfs if the factorization of the trivial span $1_* : * \to *, *$ is trivial:

- it is generated by a weakly left transitive, weakly right transitive, weakly left connected, weakly right connected, functorial reflexive relation $X \to \Gamma(X) \to X, X$.
- it is type-theoretic: (1) all objects are fibrant and (2) the two-sided Frobenius condition holds.

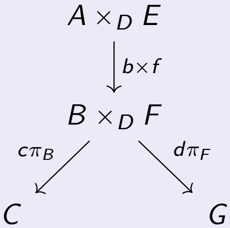
Type-theoretic 2SWFSs

Two-sided Frobenius condition.

The two-sided Frobenius condition holds when for any 'composable' two rooted cofibrations where db is a future fibration and $d'f$ is a past fibration,



the 'composite' is a cofibration.



Question

How do we algebraize this?

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Relative (co)monads

- The functor that takes a span to its right factor is a monad (Street).

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- What about the left part? Should it produce a sprout from a span?

Relative (co)monads

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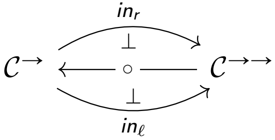
- What about the left part? Should it produce a sprout from a span?
- We consider a *relative* monad.

Back to weak factorization systems

We have the following adjunctions

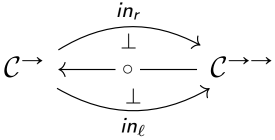


where in_r, in_ℓ are sections of $\circ$.

- A *factorization* is a section F of $\circ$.

Back to weak factorization systems

We have the following adjunctions



where in_r, in_ℓ are sections of $\circ$.

- A *factorization* is a section F of $\circ$.
- Algebraic wfs ask that F underlies (co)monads.

Back to weak factorization systems

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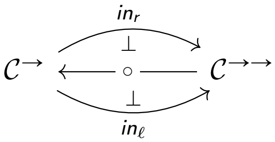


where in_r, in_ℓ are sections of $\circ$.

- A *factorization* is a section F of $\circ$.
- Algebraic wfs ask that F underlies (co)monads.
- Instead, we ask that F underlies a monad relative to in_r and a comonad relative to in_l.
- F underlies such a relative monad if given a morphism of the form $in_r f \to Fg$, we can produce a morphism of the form $Ff \to Fg$ satisfying axioms.

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ \| & & \downarrow\lambda g \\ \bullet & \xrightarrow{b} & \bullet \\ \downarrow f & & \downarrow \rho g \\ \bullet & \xrightarrow{c} & \bullet \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ \downarrow\lambda f & & \downarrow\lambda g \\ \bullet & \xrightarrow{b'} & \bullet \\ \downarrow \rho f & & \downarrow \rho g \\ \bullet & \xrightarrow{c} & \bullet \end{array}$$

Back to weak factorization systems

- A *fibration* is a composable pair $(f, g) \in \mathcal{C}^{\to\to}$ such that for any morphism $in_r z \to (f, g)$ there is a morphism $Fz \to (f, g)$ satisfying axioms.

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ \| & & \downarrow f \\ \bullet & \xrightarrow{b} & \bullet \\ \downarrow z & & \downarrow g \\ \bullet & \xrightarrow{c} & \bullet \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ \downarrow \lambda z & & \downarrow f \\ \bullet & \xrightarrow{b'} & \bullet \\ \downarrow \rho z & & \downarrow g \\ \bullet & \xrightarrow{c} & \bullet \end{array}$$

- Dually for cofibrations.
- Fibrations lift agains cofibrations.

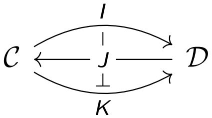
Back to weak factorization systems

Theorem (vdB-M-N)

- An algebraic weak factorization system produces a *relative* weak factorization system.
- Given such a rwfs, a pair of the form $(1, f)$ is a fibration if and only if f is an algebra of the pointed endofunctor of the algebraic weak factorization system.

Relative weak factorization systems

In general, given a pair of adjunctions

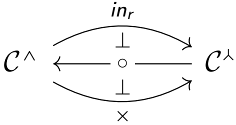


where I and K are sections of J, a relative weak factorization system is a section of J which underlies a monad relative to I and a comonad relative to K.

- Cofibrations lift against fibrations.

Two-sided weak factorization systems

We have the following adjunctions



where in_r and $\times$ are sections of $\circ$.

- Rwfs in this setting correspond to 2swfs.
- The fibrations of the form $(1; f, g)$ correspond to two-sided fibrations.
- Cofibrations correspond to rooted cofibrations.

Thank you!