

# The effective model structure on simplicial objects

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# Background

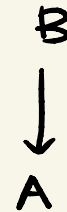
Voevodsky (2006) : model of Martin-Löf type theory

in  $\underline{\text{SSet}} = [\Delta^{\text{op}}, \underline{\text{Set}}]$

IDEA

$x : A \vdash B(x) : \text{type}$

$\rightsquigarrow$



Kan fibration

REMARK

- Univalence Axiom is valid
- Model is defined using a non-constructive metatheory (ZFC + 2 inaccessible cardinals).

## PROBLEM

Is there a constructive version of the simplicial model of Univalent Foundations?

## TWO STRANDS

- ① Cubical:  $[\square^{\text{op}}, \text{Set}]$  (Coquand & collaborators, ...)
- ② Simplicial (van den Berg & Faber, G. & Settler, ...)

Note Algebraic notion of fibration important for

both ① & ②.

Theorem [H] Constructively, SSet admits a Quillen model structure  $(W, \text{Fib}, \text{Cof})$ , where

• Fib = Kan fibrations

• Cof =  $\{ i: A \rightarrow B \mid (\forall n) i_n: A_n \rightarrow B_n \text{ is a decidable inclusion \& condition on degeneracy} \}$ .

NOTE Not every object is cofibrant.

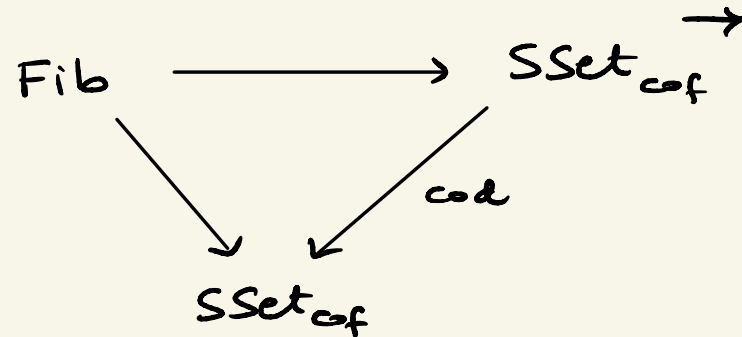
[H] S. Henry, "A constructive treatment of the Kan-Quillen model structure and of Kan's  $Ex^\infty$  functor",

ArXiv, 2019.



# Towards constructive univalence

Theorem [GH] There is a comprehension category



pseudo-stable  $\Sigma$ -types, partially pseudo-stable Id-types,  
weakly stable  $\Pi$ -types and a pseudo Tarski universe  
 $\bar{U}_c \rightarrow U_c$  that is a univalent fibration.

[GH] N. G. & S. Henry, "Towards a constructive simplicial model of univalent foundations", J LMS 2022

## Constructive Kan-Quillen model structure: two other proofs

- ① using Kan's  $\text{Ex}^\infty$  functor & standard techniques of homotopy theory.
- ② using Frobenius and Equivalence Extension property (cf. [S] Sattler's "The equivalence extension property and model categories").

[GSS] N. G., C. Sattler, K. Szumito, "The constructive

Kan-Quillen model structure: two new proofs"

(Q Journal of Math, 2022)

RECALL  $[H]$  and  $[GSS]$  work with

$$[\Delta^{\text{op}}, \underline{\text{Set}}]$$

the category of sets of CZF (Aczel)

IDEA Generalise to

$$[\Delta^{\text{op}}, \mathcal{E}]$$

a sufficiently good category (e.g. a Grothendieck topos)

QUESTION How far can you weaken the assumptions

on  $\mathcal{E}$  to carry over  $[H]$  or  $[GSS]$ ?

## The effective model structure

Theorem [GHSS] Let  $\mathcal{E}$  be countably lex extensive category. Then  $\mathcal{S}\mathcal{E} = [\Delta^{\text{op}}, \mathcal{E}]$  admits a Quillen model structure  $(\underline{W}, \underline{\text{Cof}}, \underline{\text{Fib}})$ , where

$$\underline{\text{Fib}} = \{ X \xrightarrow{f} Y \mid \forall E \in \mathcal{E},$$

$$\text{Hom}(E, X) \xrightarrow{\text{Hom}(E, f)} \text{Hom}(E, Y) \text{ Kan fibration in } \underline{\text{Set}} \}.$$

[GHSS] N.G., S. Henry, C. Sattler, K. Szumito, "The effective model structure and  $\omega$ -groupoid objects"

Forum of Mathematics, Sigma, 2022.

## OUTLINE OF THE TALK

- ① The constructive Kan - Quillen model structure : outline of one proof
- ② The effective model structure : some aspects.

# ① The constructive Kan-Quillen model structure

Set : the category of sets of CZF  $\left\{ \begin{array}{l} \cdot \text{ complete} \\ \cdot \text{ cocomplete} \\ \cdot \text{ lccc} \end{array} \right. \dots$

Definition A map  $i: A \rightarrow B$  is a **decidable inclusion** if

there is  $j: C \rightarrow B$  such that

$$[i, j]: A + C \xrightarrow{\cong} B$$

Proposition Set admits a wfs

(decidable inclusions, split surjections).

It is cofibrantly generated by  $\{ \emptyset \rightarrow 1 \}$ .

Let  $\underline{sSet} = [\Delta^{op}, \underline{Set}]$ .

$$\mathcal{I} = \{ \partial \Delta[n] \longrightarrow \Delta[n] \}, \quad \mathcal{J} = \{ \Lambda^k[n] \longrightarrow \Delta[n] \}$$

Define

$$\bullet \quad \underline{TrivFib} = \mathcal{I}^{\triangleright}, \quad \underline{Cof} = \triangleright(\underline{TrivFib})$$

$$\bullet \quad \underline{Fib} = \mathcal{J}^{\triangleright}, \quad \underline{TrivCof} = \triangleright(\underline{Fib})$$

Fact  $\underline{TrivCof} \subseteq \underline{Cof}$ ,  $\underline{TrivFib} \subseteq \underline{Fib}$

NOTE 'Existence' is understood in a strong sense.

## Proposition

(i) (Cof, Triv Fib) forms a wfs on sSet.

(ii) (Triv Cof, Fib) forms a wfs on sSet.

Proof: minor variant of small object argument.

PLAN Introduce a class  $W$  of 'weak equivalences' that satisfies 3-for-2 and such that

$$W \cap \underline{\text{Cof}} = \underline{\text{Triv Cof}}$$

§

acyclic cofibrations

$$W \cap \underline{\text{Fib}} = \underline{\text{Triv Fib}}$$

§

acyclic fibrations



## STEP 1 : Understanding Cof

Proposition The following wfs on sSet coincide:

(i) (Cof, TrivFib)

(ii) The wfs on sSet induced by the wfs  
(decidable inclusions, split surjections) on  
Set à la Reedy.

Proposition A map  $i: A \rightarrow B$  is in Cof iff

(i)  $i: A \rightarrow B$  is levelwise decidable inclusion

(ii)  $\forall$  degeneracy  $[m] \rightarrow [n]$ ,  $A_m \cup_{A_n} B_n \rightarrow B_m$  is

decidable inclusion.

## STEP 2 : The fibration category

Proposition The category of cofibrant Kan complexes

a fibration category structure, where

- weak equivalence = homotopy equivalences.
- fibrations = Kan fibrations.

Lemme Let  $X, Y$  be cofibrant Kan complexes. Then  $f: X \rightarrow Y$  is a trivial Kan fibration if and only if it is an acyclic Kan fibration.

### STEP 3 The restricted Frobenius property

Proposition Let  $f: X \rightarrow Y$  be a fibration with

$X$  cofibrant. Then

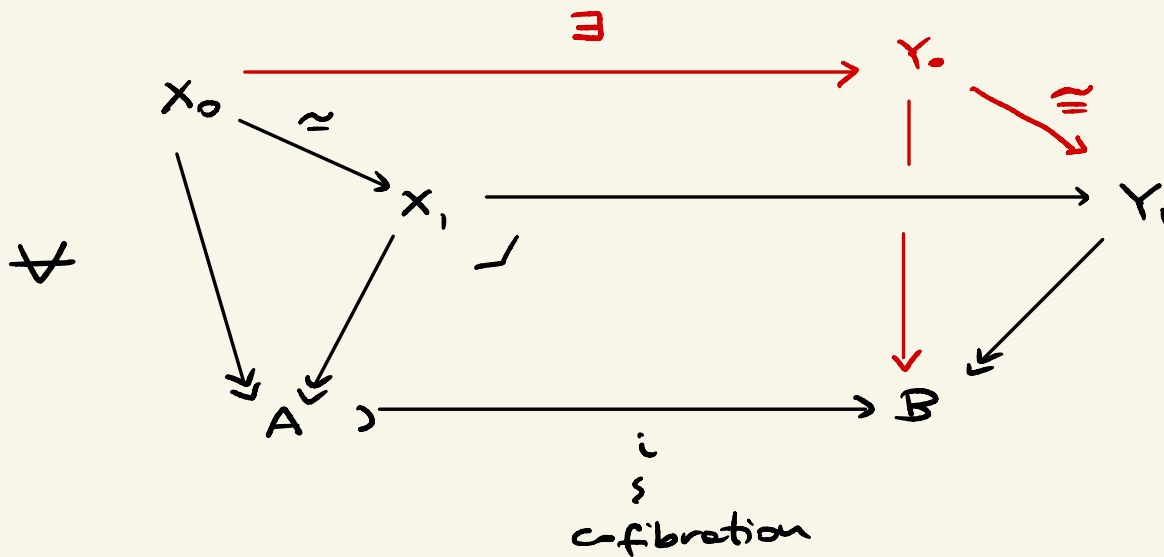
$$f^*: \underline{sSet}/Y \longrightarrow \underline{sSet}/X .$$

preserves trivial cofibrations

Proof Similar to [GS], but with additional cofibrancy considerations.

STEP 4 The equivalence extension property

Proposition In  $\underline{sSet}_{\text{cof}}$ , we have



Proof Use  $i^* + \pi_i$  as  $[s]$ , but need to use that  $\pi_i$  preserves cofibrant objects (!) to remain in  $\underline{sSet}_{\text{cof}}$

## STEP 5 : The model structure on SSet<sub>cof</sub>

Extend the notion of weak equivalence from cofibrant Kou complexes to sSet<sub>cof</sub> via fibrant replacement

Lemma In sSet<sub>cof</sub>

- (i) Let  $f: X \rightarrow Y$  be a Kou fibration. Then  $f$  is a trivial Kou fibration iff it is a weak equivalence.
- (ii) Let  $i: A \rightarrow B$  be a cofibration. Then  $i$  is a trivial cofibration iff it is a weak equivalence.

Thm  $(\underline{W}_{\text{cof}}, \underline{\text{Cof}}_{\text{cof}}, \underline{\text{Fib}}_{\text{cof}})$  is a Quillen model structure on sSet<sub>cof</sub>.

## The model structure

Extend the notion of weak equivalence from SSet<sub>cof</sub> to SSet via cofibrant replacement

Lemma In SSet

(i) Let  $f: X \rightarrow Y$  be a Kan fibration. Then  $f$  is a trivial Kan fibration iff it is a weak equivalence

(ii) Let  $i: A \rightarrow B$  be a cofibration. Then  $i$  is a trivial cofibration iff it is a weak equivalence.

Thm  $(\underline{W}, \underline{Cof}, \underline{Fib})$  is a Quillen model

structure on SSet.

## ② The effective model structure

Let  $\mathcal{E}$  be a countably lex extensive category, i.e.

- $\mathcal{E}$  has finite limits
- $\mathcal{E}$  has countable coproducts
- Countable coproducts in  $\mathcal{E}$  are preserved by pullback and are disjoint.

### NOTE

- No arbitrary colimits
- No local cartesian closure

Remark In  $\mathcal{E}$ , we have a wfs of decidable inclusions and split epimorphisms.

Obs If  $S \in \underline{\text{Set}}$  is countable, we have

$$\underline{S} = \bigsqcup_{s \in S} 1 \in \mathcal{E}$$

Similarly, for  $K \in \underline{s\text{Set}}$  levelwise countable, we have

$$\underline{K} \in \underline{s\mathcal{E}}$$



# Fibrations

Let

$$I_{sE} = \{ \underline{\partial \Delta[n]} \rightarrow \underline{\Delta[n]} \}$$

$$J_{sE} = \{ \underline{\Lambda^k[n]} \rightarrow \underline{\Delta[n]} \}$$

Define

$$\underline{\text{Triv Fib}} = \uparrow I_{sE}$$

$$\underline{\text{Cof}} = \uparrow \underline{\text{Triv Fib}}$$

$$\underline{\text{Fib}} = \uparrow J_{sE}$$

$$\underline{\text{Triv Cof}} = \uparrow \underline{\text{Fib}}$$

NOTE

These are **enriched** lifting properties.

# Evaluation

$$K \in \underline{sSet} \text{ finite, } X \in sE \Rightarrow X(K) = \int_{[n] \in \Delta} X_n^{K_n} \in E$$

Examples:  $X(\Delta[n]) = X_n$ ,  $X(\Lambda^k[n]) = X_1 \times_{X_0} X_1$

sSet

$\Lambda^k[n]$

$i \downarrow$

$\Delta[n]$

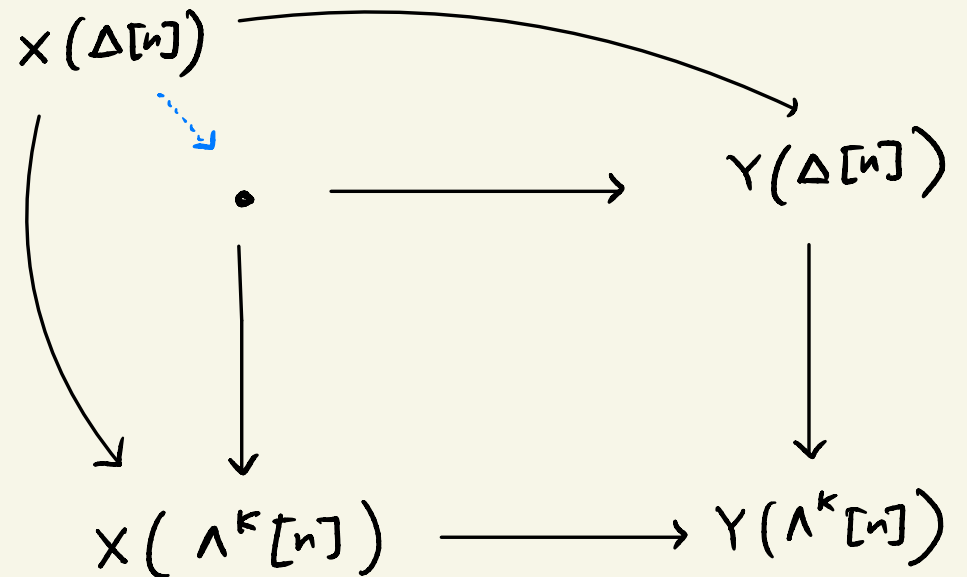
sE

$X$

$\downarrow P$

$Y$

$\Rightarrow$



Proposition Let  $f: X \rightarrow Y$  in  $\mathcal{E}$ . TFAE

(i)  $p \in \underline{\text{Fib}}$

(ii) For every  $\Lambda^k[n] \rightarrow \Delta[n]$ , the map

$$X(\Delta[n]) \longrightarrow X(\Lambda^k[n]) \times_{Y(\Lambda^k[n])} Y(\Delta[n])$$

is split epi in  $\mathcal{E}$ .

(iii)

$$\begin{array}{ccc}
 \Lambda^k[n] & \longrightarrow & \text{Hom}(E, X) \\
 \downarrow & & \downarrow \\
 \Delta[n] & \longrightarrow & \text{Hom}(E, Y)
 \end{array}$$

has diagonal filler  
 $\forall E \in \mathcal{E}$

## Proposition

- (i)  $(\underline{\text{Cof}}, \underline{\text{Triv Fib}})$  forms an enriched wfs on  $\mathcal{SE}$ ,
- (ii)  $(\underline{\text{Triv Cof}}, \underline{\text{Fib}})$  forms an enriched wfs on  $\mathcal{SE}$ .

Remark This requires keeping track of inclusions that are complemented, to be able to avoid assumption of cocompleteness.

## NEXT STEPS

STEP 1 (Understanding cof) : delicate, cannot directly use Reedy theory (as we do not assume all small colimits).

STEP 2 (Fibration category) OK, uses only finite limits

STEP 3 (Frobenius) Similar

STEP 4 (Equivalence extension property) : Need to define (!)  $\Pi_i$ , as no lccc is assumed

STEP 5 / End : Similar, very formal

## Further aspects

- [GHSS] also studies  $H_{\infty}(sE)$
- Some connection to theory of exact completions.
- Remark  $E$  Grothendieck topos  ~~$\Rightarrow$~~   
 $H_{\infty}(sE)$  Grothendieck  $\infty$ -topos
- Next : Avoid countable coproducts, use  $\mathbb{N}$ .