The effective model structure on simplicial objects

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Background

Voevodsky (2006) : model of Martin-Löf type theory

in $\underline{\mathrm{sSet}} = [\Delta^{op}, \underline{\mathrm{Set}}]$

IDEA $x : A \vdash B(x) : \mathrm{type}$ $\rightsquigarrow$ $\begin{array}{c} B \\ \downarrow \\ A \end{array}$ Kan fibration

REMARK

- Univalence Axiom is valid
- Model is defined using a non-constructive metatheory (ZFC + 2 inaccessible cardinals).

PROBLEM

Is there a constructive version of the simplicial model of Univalent Foundations ?

TWO STRANDS

1. Cubical: $[\square^{op}, Set]$ (Coquand & collaborators, ...)

2. Simplicial (van den Berg & Faber, G. & Sattler, ...)

Note Algebraic notion of fibration important for both 1 & 2.

Theorem [H] Constructively, $\underline{\mathrm{sSet}}$ admits a Quillen model structure $(W, \mathrm{Fib}, \mathrm{Cof})$, where

- $\underline{\mathrm{Fib}}$ = Kan fibrations
- $\underline{\mathrm{Cof}} = \{ i : A \longrightarrow B \mid (\forall n)\ i_n : A_n \longrightarrow B_n$ is a decidable inclusion & condition on degeneracy $\}$.

Note Not every object is cofibrant.

[H] S. Henry, "A constructive treatment of the Kan-Quillen model structure and of Kan's Ex^{∞} functor", ArXiv, 2019.

Towards constructive univalence

Theorem [GH] There is a comprehension category

$$\begin{tikzcd} \mathrm{Fib} \arrow[rr] \arrow[dr] & & \mathrm{SSet}_{cof}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathrm{SSet}_{cof} & \end{tikzcd}$$

pseudo-stable Σ-types, partially pseudo-stable Id-types, weakly stable Π-types and a pseudo Tarski universe $\tilde{U}_c \longrightarrow U_c$ that is a univalent fibration.

[GH] N. G. & S. Henry, "Towards a constructive simplicial model of univalent foundations", J LMS 2022

Constructive Kan-Quillen model structure: two other proofs

1. using Kan's Ex^∞ functor & standard techniques of homotopy theory.

2. using Frobenius and Equivalence Extension property (cf. [S] Sattler's "The equivalence extension property and model categories").

[GSS] N. G, C. Sattler, K. Szumiło, "The constructive Kan-Quillen model structure: two new proofs" (Q Journal of Math, 2022)

RECALL [H] and [GSS] work with

$$[\Delta^{op}, \underline{\mathrm{Set}}]$$

the category of sets of CZF (Aczel)

IDEA Generalise to

$$[\Delta^{op}, \mathcal{E}]$$

a sufficiently good category (e.g. a Grothendieck topos)

QUESTION How far can you weaken the assumptions on $\mathcal{E}$ to carry over [H] or [GSS] ?

The effective model structure

Theorem [GHSS] Let $\mathcal{E}$ be countably lextensive category. Then $s\mathcal{E} = [\Delta^{op}, \mathcal{E}]$ admits a Quillen model structure $(\underline{W}, \underline{Cof}, \underline{Fib})$, where

$$\underline{Fib} = \{ X \xrightarrow{f} Y \mid \forall E \in \mathcal{E},$$

$$\mathrm{Hom}(E, X) \xrightarrow{\mathrm{Hom}(E,f)} \mathrm{Hom}(E, Y) \text{ Kan fibration in } \underline{SSet} \}.$$

[GHSS] N. G, S. Henry, C. Sattler, K. Szumito, "The effective model structure and ∞-groupoid objects" Forum of Mathematics, Sigma, 2022.

OUTLINE OF THE TALK

1. The constructive Kan-Quillen model structure: outline of one proof

2. The effective model structure: some aspects.

① The constructive Kan-Quillen model structure

Set : the category of sets of CZF $\begin{cases} \bullet \text{ complete} \\ \bullet \text{ cocomplete} \\ \bullet \text{ lccc} \end{cases}$ ...

Definition A map $i : A \longrightarrow B$ is a decidable inclusion if there is $j : C \longrightarrow B$ such that

$$[i, j] : A + C \xrightarrow{\cong} B$$

Proposition **Set** admits a wfs

(decidable inclusions, split surjections).

It is cofibrantly generated by $\{ \emptyset \longrightarrow 1 \}$.

Let $\underline{sSet} = [\Delta^{op}, \underline{Set}]$.

$$I = \{ \partial\Delta[n] \longrightarrow \Delta[n] \}, \qquad J = \{ \Lambda^k[n] \longrightarrow \Delta[n] \}$$

Define

- $\underline{TrivFib} = I^{\pitchfork}$, $\qquad \underline{Cof} = {}^{\pitchfork}(TrivFib)$
- $\underline{Fib} = J^{\pitchfork}$, $\qquad \underline{TrivCof} = {}^{\pitchfork}(\underline{Fib})$

<u>Fact</u> $\quad \underline{TrivCof} \subseteq \underline{Cof}$, $\quad \underline{TrivFib} \subseteq \underline{Fib}$

<u>NOTE</u> $\quad$ 'Existence' is understood in a strong sense.

Proposition

(i) $(\underline{\mathrm{Cof}}, \underline{\mathrm{TrivFib}})$ forms a wfs on $s\underline{\mathrm{Set}}$.

(ii) $(\underline{\mathrm{TrivCof}}, \underline{\mathrm{Fib}})$ forms a wfs on $s\underline{\mathrm{Set}}$

Proof : minor variant of small object argument.

PLAN Introduce a class W of 'weak equivalences' that satisfies 3-for-2 and such that

$$\underbrace{W \cap \underline{\mathrm{Cof}}}_{\text{acyclic cofibrations}} = \underline{\mathrm{TrivCof}} \qquad \underbrace{W \cap \underline{\mathrm{Fib}}}_{\text{acyclic fibrations}} = \underline{\mathrm{TrivFib}}$$

STEP 1 : Understanding $\underline{\mathrm{Cof}}$

Proposition The following wfs on $\underline{\mathrm{sSet}}$ coincide:

(i) $(\underline{\mathrm{Cof}}, \underline{\mathrm{TrivFib}})$

(ii) The wfs on $\underline{\mathrm{sSet}}$ induced by the wfs (decidable inclusions, split surjections) on $\underline{\mathrm{Set}}$ à la Reedy.

Proposition A map $i : A \longrightarrow B$ is in $\underline{\mathrm{Cof}}$ iff

(i) $i : A \longrightarrow B$ is levelwise decidable inclusion

(ii) $\forall$ degeneracy $[m] \longrightarrow [n]$, $A_m \sqcup_{A_n} B_n \longrightarrow B_m$ is decidable inclusion.

STEP 2 : The fibration category

Proposition The category of cofibrant Kan complexes has a fibration category structure, where

- weak equivalence = homotopy equivalences.
- fibrations = Kan fibrations.

Lemma Let X, Y be cofibrant Kan complexes. Then $f : X \longrightarrow Y$ is a trivial Kan fibration if and only if it is an acyclic Kan fibration.

STEP 3 The restricted Frobenius property

Proposition Let $f : X \longrightarrow Y$ be a fibration with X cofibrant. Then

$$f^* : \underline{sSet}/Y \longrightarrow \underline{sSet}/X .$$

preserves trivial cofibrations

Proof Similar to [GS], but with additional cofibrancy considerations.

STEP 4 The equivalence extension property

Proposition In $\underline{sSet}_{cof}$, we have

$$\forall \quad \begin{tikzcd} X_0 \arrow[rr, "\exists"] \arrow[dr, "\simeq"] \arrow[dd, two heads] & & Y_0 \arrow[dr, "\cong"] \arrow[dd, two heads] & \\ & X_1 \arrow[rr] \arrow[dl, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & Y_1 \arrow[dl, two heads] \\ A \arrow[rr, hook, "i"'] & & B & \end{tikzcd}$$

i cofibration

Proof Use $i^* \dashv \Pi_i$ as [S], but need to use that Π_i preserves cofibrant objects (!) to remain in $\underline{sSet}_{cof}$

STEP 5 : The model structure on $\underline{\mathrm{sSet}}_{cof}$

Extend the notion of weak equivalence from cofibrant Kan complexes to $\underline{\mathrm{sSet}}_{cof}$ via fibrant replacement

Lemma In $\underline{\mathrm{sSet}}_{cof}$

(i) Let $f : X \longrightarrow Y$ be a Kan fibration. Then f is a trivial Kan fibration iff it is a weak equivalence.

(ii) Let $i : A \longrightarrow B$ be a cofibration. Then i is a trivial cofibration iff it is a weak equivalence.

Thm $(\underline{W}_{cof}, \underline{\mathrm{Cof}}_{cof}, \underline{\mathrm{Fib}}_{cof})$ is a Quillen model structure on $\underline{\mathrm{sSet}}_{cof}$.

The model structure

Extend the notion of weak equivalence from $\underline{\mathrm{sSet}}_{cof}$ to $\underline{\mathrm{sSet}}$ via cofibrant replacement

Lemma In $\underline{\mathrm{sSet}}$

(i) Let $f : X \longrightarrow Y$ be a Kan fibration. Then f is a trivial Kan fibration iff it is a weak equivalence

(ii) Let $i : A \longrightarrow B$ be a cofibration. Then i is a trivial cofibration iff it is a weak equivalence.

Thm $(\underline{W}, \underline{\mathrm{Cof}}, \underline{\mathrm{Fib}})$ is a Quillen model structure on $\underline{\mathrm{sSet}}$.

2 The effective model structure

Let $\mathcal{E}$ be a countably lextensive category, ie.

- $\mathcal{E}$ has finite limits
- $\mathcal{E}$ has countable coproducts
- Countable coproducts in $\mathcal{E}$ are preserved by pullback and are disjoint.

NOTE

- No arbitrary colimits
- No local cartesian closure

Remark In $\mathcal{E}$, we have a wfs of decidable inclusions and split epimorphisms.

Obs If $S \in \underline{\mathrm{Set}}$ is countable, we have

$$\underline{S} = \coprod_{s \in S} 1 \in \mathcal{E}$$

Similarly, for $K \in \underline{s\mathrm{Set}}$ levelwise countable, we have

$$\underline{K} \in s\mathcal{E}$$

Fibrations

Let

$$I_{SE} = \{ \underline{\partial\Delta[n]} \longrightarrow \underline{\Delta[n]} \} \qquad J_{SE} = \{ \underline{\Lambda^k[n]} \longrightarrow \underline{\Delta[n]} \}$$

Define

$$\underline{\text{Triv Fib}} = {}^{\pitchfork} I_{SE} \qquad \underline{\text{Cof}} = {}^{\pitchfork}\, \underline{\text{Triv Fib}}$$

$$\underline{\text{Fib}} = {}^{\pitchfork} J_{SE} \qquad \underline{\text{Triv Cof}} = {}^{\pitchfork}\, \underline{\text{Fib}}$$

NOTE These are enriched lifting properties.

Evaluation

$$K \in \underline{sSet} \text{ finite}, \quad X \in s\mathcal{E} \quad \Longrightarrow \quad X(K) = \int_{[n]\in\Delta} X_n^{K_n} \in \mathcal{E}$$

Examples: $X(\Delta[n]) = X_n$, $\quad X(\Lambda^k[n]) = X_1 \times_{X_0} X_1$

$$\begin{tikzcd} \underline{sSet} & \underline{s\mathcal{E}} \\ \Lambda^k[n] \arrow[d, "i"'] & X \arrow[d, "p"] \\ \Delta[n] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X(\Delta[n]) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \bullet \arrow[r] \arrow[d] & Y(\Delta[n]) \arrow[d] \\ & X(\Lambda^k[n]) \arrow[r] & Y(\Lambda^k[n]) \end{tikzcd}$$

Proposition Let $f : X \longrightarrow Y$ in $s\mathcal{E}$. TFAE

(i) $p \in \underline{\mathrm{Fib}}$

(ii) For every $\Lambda^k[n] \longrightarrow \Delta[n]$, the map

$$X(\Delta[n]) \longrightarrow X(\Lambda^k[n]) \times_{Y(\Lambda^k[n])} Y(\Delta[n])$$

is split epi in $\mathcal{E}$.

(iii)
$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{Hom}(E, X) \arrow[d] \\
\Delta[n] \arrow[r] & \mathrm{Hom}(E, Y)
\end{tikzcd}$$
has diagonal filler $\forall E \in \mathcal{E}$

Proposition

(i) $(\underline{Cof}, \underline{TrivFib})$ forms an enriched wfs on $s\mathcal{E}$,

(ii) $(\underline{TrivCof}, \underline{Fib})$ forms an enriched wfs on $s\mathcal{E}$.

Remark This requires keeping track of inclusions that are complemented, to be able to avoid assumption of cocompleteness.

NEXT STEPS

STEP 1 (Understanding $\mathcal{C}of$) : delicate, cannot directly use Reedy theory (as we do not assume all small colimits).

STEP 2 (Fibration category) OK, uses only finite limits

STEP 3 (Frobenius) Similar

STEP 4 (Equivalence extension property) : Need to define (!) Π_i, as no lccc is assumed

STEP 5 / End : Similar, very formal

Further aspects

- [GHSS] also studies $\mathrm{Ho}_\infty(s\mathcal{E})$

- Some connection to theory of exact completions.

- <u>Remark</u> $\mathcal{E}$ Grothendieck topos $\not\Rightarrow$ $\mathrm{Ho}_\infty(s\mathcal{E})$ Grothendieck ∞-topos

- <u>Next</u> : Avoid countable coproducts, use $\mathbb{N}$.