Monads on Categories of Relational Structures

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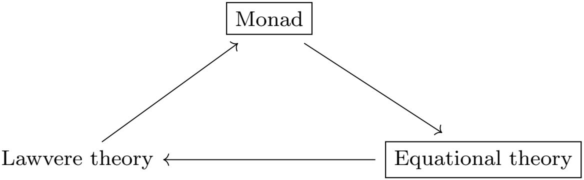
Background

- E. Moggi (1991): *Computational effects* as monads/Kleisli triples
  e.g. categorical semantics of (probabilistic) non-determinism
- G. Plotkin & J. Power (2001): *algebraic effects*
  - computational effects arise from *operations and equations*
  - based on connection between monads and algebraic theories

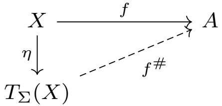


Equational theories

- **Signature** Σ: operation symbols σ with assigned arities $\mathsf{ar}(\sigma) \in \mathbb{N}$
- **Σ-algebra**: set A equipped with functions

$$\sigma_A \colon A^n \to A \qquad\qquad \text{equivalently: } \mathsf{Set}(n, A) \to A$$

- Σ-algebras and *homomorphisms* form a category $\mathsf{Alg}(\Sigma)$
- The free algebra of Σ-terms on X:



- **Varieties**: full subcategories $\mathsf{Alg}(\mathbb{T}) \hookrightarrow \mathsf{Alg}(\Sigma)$ specified by a set $\mathbf{T}$ of equations

$$A \models s = t \;\text{ if } f^{\#}(s) = f^{\#}(t) \text{ for all } f \colon \mathsf{Vars} \to A$$

Equational theories and monads

Theorem

Every finitary monad on Set is the free algebra monad of an equational theory. Moreover, $\mathsf{Alg}(\mathbf{T}) \cong \mathsf{Alg}(M_{\mathbf{T}})$ (as concrete categories).

- $TX = T_{\Sigma}(X)$ modulo derivable equality in the equational logic of $\mathbf{T}$
- $\eta \colon X \to TX$ is "inclusion of variables as terms"
- $\mu \colon TTX \to TX$ is given by the "flattening" of complex terms

Monad-theory correspondences

- G. Kelly and J. Power (1993): presentations of enriched finitary monads
  - ▷ Key idea I: arities of operations = finitely presentable objects
  - ▷ Key idea II: structured signatures $\rightsquigarrow$ equational presentations
- Recent syntactic accounts of monads beyond Set, e.g.
  - ▷ J. Adámek, C. Ford, S. Milius, L. Schröder (2020):

    finitary (enriched) monads on Pos = inequational theories

  - ▷ R. Mardare, P. Panangaden, G. Plotkin (2016):

    quantitative algebraic theories $\hookrightarrow$ monads on Met

- Categories of Relational Structures -

Overview

Slogan: Horn theories balance expressive power with 'nice' categorical structure.

Power

- Set: sets/functions
- Pos: posets/monotone maps
- Met: metric spaces/nonexpansive maps
- Par: partial algebras/homomorphisms

Structure

- locally presentable categories
- closed monoidal structure

Relational structures

- **Relational signature** Π: relation symbols α with finite arities $\mathsf{ar}(\alpha) \in \mathbb{N}$
- **Π-structure**: set X equipped with a set $\mathsf{E}(X)$ of edges $(\alpha, f \colon \mathsf{ar}(\alpha) \xrightarrow{f} X)$
- $\mathsf{Str}(\Pi)$: category of Π-structures with relation-preserving maps

$$h \colon X \to Y, \qquad X \models \alpha(f) \text{ implies } Y \models \alpha(h \cdot f)$$

- **Gra** $= \mathsf{Str}(\Pi)$ for $\Pi = \{\leq\}$:

$$(\leq, f \colon \{0,1\} \to X) \in \mathsf{E}(X) \leftrightsquigarrow X \models f(0) \leq f(1)$$

Horn theories

- **Horn sentence over** Π: expressions $\Phi \implies \psi$ where
  - ▷ Φ is a set of Π-atoms (i.e. expressions $R(x_1, \ldots, x_n)$)
  - ▷ ψ is a $\Pi \sqcup \{=\}$-atom
- $\Phi \implies \psi$ is **λ-ary** if λ is a regular cardinal with $\mathsf{card}\,\Phi < \lambda$
- These are universal sentences of the infinitary logic $\mathbf{L}_{\lambda,\lambda}$:

$$\boxed{x \leq y, y \leq x \implies x = y} \text{ encodes } \boxed{\forall x, y.(x \leq y \land y \leq x \to x = y)}$$

- Write $\mathscr{H} = (\Pi, \mathcal{A})$ where $\mathcal{A}$ is a set of λ-ary Horn sentences

We work with the full subcategory $\mathsf{Str}(\mathscr{H}) \hookrightarrow \mathsf{Str}(\Pi)$ of $\mathscr{H}$-models

Examples

- $\mathsf{Pos} = \mathsf{Str}(\mathscr{H})$ for the ω-ary theory $\mathscr{H}$ with $\Pi = \{\leq\}$ and axioms

$$\top \implies x \leq x \qquad \{x \leq y, y \leq z\} \implies x \leq z \qquad \{x \leq y, y \leq x\} \implies x = y$$

- $\mathsf{Met} \cong \mathsf{Str}(\mathscr{H})$ (as concrete categories) for an ω_1-ary Horn theory
  - ▷ Π has binary relations $\sim_\epsilon$ for all $\epsilon \in \mathbb{Q} \cap [0,1]$
  - ▷ interpret $X \models x \sim_\epsilon y$ as $d(x,y) \leq \epsilon$:

$$d(x,y) := \bigwedge\{\epsilon \in \mathbb{Q} \cap [0,1] \mid X \models x \sim_\epsilon y\}$$

  - ▷ emphasis: this requires an ω_1-ary axiom

$$\{x \sim_\delta y \mid \mathbb{Q} \cap [0,1] \ni \delta > \epsilon\} \implies x \sim_\epsilon y$$

Local presentability

Proposition

$\mathsf{Str}(\mathscr{H})$ is a full (epi-)reflective subcategory of $\mathsf{Str}(\Pi)$ closed under λ-directed colimits.

- The embedding $\mathsf{Str}(\Pi, \mathcal{A}) \hookrightarrow \mathsf{Str}(\Pi)$ has a left adjoint

$$\mathsf{Str}(\Pi) \xrightarrow{R} \mathsf{Str}(\Pi, \mathcal{A}) \qquad \text{(the } \textit{reflector}\text{)}$$

- Consequence: $\mathsf{Str}(\mathscr{H})$ is **locally λ-presentable**:
  - ▷ $\mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$ is essentially small ($\mathscr{C}(X, -) \colon \mathscr{C} \to \mathsf{Set}$ is λ-accessible)
  - ▷ each $X \in \mathsf{Str}(\mathscr{H})$ is a λ-directed colimit of λ-presentable objects
- $X \in \mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$ iff $X \cong R(Y)$ for some $Y \in \mathsf{Pres}_\lambda(\mathsf{Str}(\Pi))$
  - ▷ $\mathsf{Str}(\Pi) \rightsquigarrow \mathsf{card}\, X, \mathsf{card}\, \mathsf{E}(X) < \lambda$
  - ▷ $\mathsf{Pos} \rightsquigarrow$ finite posets
  - ▷ $\mathsf{Met} \rightsquigarrow$ countable spaces

Closed monoidal structure

- Let $[X, Y]$ denote the Π-structure on $\mathsf{Str}(\mathscr{H})(X, Y)$ defined point-wise:

$$[X, Y] \models \alpha(f_1, \ldots, f_n) :\iff Y \models \alpha(f_1(x), \ldots, f_n(x)) \text{ for all } x \in X$$

- $[-, -]$ is part of a *closed (symmetric) monoidal structure* on $\mathsf{Str}(\Pi)$
- This structure inherited by $\mathsf{Str}(\mathscr{H})$ via $\mathsf{Str}(\Pi) \xrightarrow{R} \mathsf{Str}(\mathscr{H})$:
  - ▷ $X \otimes_{\mathscr{H}} Y := R(X \otimes Y)$ and $I = RI_0$
  - ▷ the Cartesian closed structure on Pos
  - ▷ the *Manhattan metric*: $(X \times Y, d)$ where

$$d((x_1, y_1), (x_2, y_2)) := \mathsf{min}(d_X(x_1, x_2) + d_Y(y_1, y_2), 1)$$

Concluding remarks

Proposition

$\mathsf{Str}(\mathscr{H})$ is locally λ-presentable as a (symmetric) monoidal closed category.

- Idea: $\mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$ is closed under $\otimes_{\mathscr{H}}$ and $I \in \mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$
- internal λ-presentable objects = external λ-presentable objects, i.e.

$$[X, -]\colon \mathsf{Str}(\mathscr{H}) \to \mathsf{Str}(\mathscr{H}) \text{ is } \lambda\text{-accessible} \qquad (X \in \mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H})))$$

- $T\colon \mathsf{Str}(\mathscr{H}) \to \mathsf{Str}(\mathscr{H})$ is **enriched** if

$$[X, Y] \models R(f_1, \ldots, f_n) \text{ implies } [TX, TY] \models R(Tf_1, \ldots, Tf_n)$$

- Relational Algebraic Theories -

Overview

Universal algebra for enriched λ-accessible monads on $\mathsf{Str}(\mathscr{H})$

- $\mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$ = internally λ-presentable objects = arities of operations
- Relations from Π afford an equations-as-relations perspective

Algebras in $\mathsf{Str}(\mathscr{H})$

- **Signature** Σ: operations equipped with $\mathsf{ar}(\sigma) \in \mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$
- **Σ-algebra**: $\mathscr{H}$-model A equipped with relation-preserving maps

$$\sigma_A \colon [\mathsf{ar}(\sigma), A] \to A$$

- **homomorphisms**: relation-preserving map $A \to B$ such that

$$\begin{array}{ccc} [\mathsf{ar}(\sigma), A] & \xrightarrow{\sigma_A} & A \\ {\scriptstyle h\cdot(-)}\downarrow & & \downarrow{\scriptstyle h} \\ [\mathsf{ar}(\sigma), B] & \xrightarrow{\sigma_B} & B \end{array} \qquad h(\sigma_A(a)) = \sigma_B(h(a))$$

$\mathsf{Alg}\,\Sigma$ denotes the category of Σ-algebras and homomorphisms

Example: Algebras in Pos

- Arities of operations = finite posets (carried by natural numbers)
- Consider $\Sigma = \{\sigma\}$ where $\mathsf{ar}(\sigma) = \mathbf{2} := (0 < 1)$
- **Σ-algebra**: poset A with monotone map $\sigma_A \colon [\mathbf{2}, A] \to A$
- Equivalently: a monotone partial map

$$\bar{\sigma} \colon A \times A \to A, \qquad \bar{\sigma}(a_0, a_1) := \sigma(f) \text{ where } f(i) = a_i$$

- $\bar{\sigma}(a, b)$ defined if $a \leq b$ in A

$\mathsf{ar}(\sigma)$ is the domain of definition of σ!

Fruit from the theory of functor algebras

- $\mathsf{Alg}(\Sigma) \cong \mathsf{Alg}(H_\Sigma)$ (as concrete categories):

$$H_\Sigma X := \coprod_{\sigma \in \Sigma} [\mathsf{ar}(\sigma), X] \qquad (\lambda\text{-accessible!})$$

- Consequences:
  - ▷ The forgetful functor $U \colon \mathsf{Alg}(\Sigma) \to \mathsf{Str}(\mathscr{H})$ is λ-accessible
  - ▷ $\mathsf{Alg}(\Sigma)$ is locally λ-presentable
  - ▷ U has a left adjoint $F \colon \mathsf{Str}(\mathscr{H}) \to \mathsf{Alg}(\Sigma)$

Relational algebraic theories

- **Σ-Terms**: least set $T_\Sigma(X) \supseteq X$ such that

$$\sigma(f) \in T_\Sigma(X) \text{ for each } \sigma \in \Sigma \text{ and } \mathit{map}\ |\mathsf{ar}(\sigma)| \xrightarrow{f} T_\Sigma(X)$$

- Variable assignments are relation-preserving $e \colon X \to A \ldots$

$$T_\Sigma(X) \xrightarrow{e^\#} A, \qquad e^\#(\sigma(s,t)) = \sigma_A(e^\#(s), e^\#(t)) \text{ possibly undefined!}$$

- **Σ-relations**: expression $\Gamma \vdash R(t_1, \ldots, t_n)$
  - ▷ $\Gamma \in \mathsf{Pres}_\lambda(\mathsf{Str}(\mathscr{H}))$
  - ▷ $R \in \Pi$ and $t_1, \ldots, t_n \in T_\Sigma(X)$

- $A \models \Gamma \vdash s \leq t$ if for every monotone $f \colon \Gamma \to A$

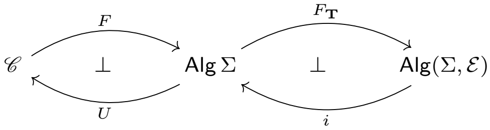
$$f^\#(s), f^\#(t) \text{ defined}; \qquad A \models f^\#(s) \leq f^\#(t)$$

From theories to monads

Theorem

There is an assignment $\mathbf{T} \mapsto M_{\mathbf{T}}$ of each relational algebraic theory $\mathbf{T}$ to an enriched λ-accessible monad $M_{\mathbf{T}}$. Moreover, $\mathsf{Alg}(\mathbf{T}) \cong \mathsf{Alg}(M_{\mathbf{T}})$.

- Σ has a presentation as a λ-accessible functor
- $\mathsf{Alg}(\mathbf{T})$ is a *reflective* subcategory of $\mathsf{Alg}\,\Sigma$ closed under λ-directed colimits
- preservation of models: Beck's monadicity theorem



The ensuing monad $M_{\mathbf{T}}$ is the *free algebra monad* of $\mathbf{T}$

Relational Logic

Sound/complete sequent calculus for relational algebraic reasoning:

$$X \vdash {\downarrow} t \ \ (\text{“definedness”}) \qquad X \vdash \alpha(t_1, \ldots, t_{\mathsf{ar}(\alpha)}) \ \ (\text{“relational”})$$

- “elimination rule for arity conditions” concludes definedness of operations:

$$\frac{\{X \vdash \alpha(f \cdot g) \mid \mathsf{ar}(\sigma) \models \alpha(g)\} \cup \{X \vdash {\downarrow} f(i) \mid i \in \mathsf{ar}(\sigma)\}}{X \vdash {\downarrow}\sigma(f)}$$

side condition: $\mathsf{ar}(\alpha) \xrightarrow{g} \mathsf{ar}(\sigma) \xrightarrow{f} T_\Sigma(X)$

Theorem

The following are equivalent:

- $X \vdash \alpha(f)$ is derivable in the relational logic of $\mathbb{T}$
- every $\mathbb{T}$-algebra satisfies $X \vdash \alpha(f)$

Free T-algebras, syntactically

- Define $FX := \{t \in T_\Sigma(X) \mid X \vdash {\downarrow}t\}$
- Quotient FX by the equivalence relation

$$s \sim t :\Longleftrightarrow X \vdash s = t \text{ is derivable}$$

- $FX/{\sim}$ has structure of $\mathscr{H}$-model with relations

$$FX/{\sim} \models \alpha(t) :\Longleftrightarrow X \vdash \alpha(t) \text{ is derivable}$$

Theorem

$FX/{\sim}$ carries the structure of a free **T**-algebra X; the universal morphism is $\eta\colon x \mapsto [x]$.

Concluding remarks

- **Relational algebraic theories**: universal algebra for monads on $\mathsf{Str}(\mathscr{H})$
  - ▷ important: $\mathsf{Str}(\mathscr{H})$ is locally presentable as a closed category
  - ▷ enrichment relates to the use of relation-preserving operations
  - ▷ Theory-to-monad direction also holds if $\kappa \leq \lambda$
- **Relational logic**: sound and complete sequent system
  - ▷ syntactic description of the free algebra monad of a theory
- **Future work** includes:
  - ▷ treatment of further enrichments
  - ▷ expand to locally presentable categories (e.g. $\mathbf{Cat}, \mathbf{Nom}, \dots$)
  - ▷ graded relational algebraic theories for *graded monads*

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