TRIPOSES AND TOPOSES THROUGH ARROW ALGEBRAS

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Introduction and motivation

Let L be a locale.

For every set I, the set L^I is a Heyting algebra under pointwise order and operations; the functor $L^{-} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytAlg}$ is a *tripos*.

Starting from L^{-} we can build a category $\mathsf{Set}[L^{-}]$ where:

- objects are pairs $(X, \sim_X)$ of a set X together with a *partial equivalence relation* $\sim_X \in L^{X \times X}$;
- morphisms $(X, \sim_X) \to (Y, \sim_Y)$ are *functional relations* $F \in L^{X \times Y}$.

Proposition (Higgs, '73)

$\mathsf{Set}[L^{-}] \simeq \mathsf{Sh}(L)$.

The previous construction is a first example of the *tripos-to-topos construction*.

Knowing that a topos can be presented as Set[P] for some tripos P allows us to reduce its *internal* logic to the *external* logic of P.

Besides localic toposes, another fundamental class of toposes arising from the tripos-to-topos construction is that of realizability toposes, built from *partial combinatory algebras* (PCAs).

For every PCA $\mathbb{A}$, the *realizability tripos* over $\mathbb{A}$ is defined by letting $P_{\mathbb{A}}(I) \coloneqq DA^I$ where DA is the set of downward-closed subsets of $\mathbb{A}$.

Example

Kleene's first model $\mathcal{K}_1$ is defined on $\mathbb{N}$ by letting $n \cdot m$ be the result of the n-th partial recursive function on input m whenever defined.

Question

How do we unify the two frameworks, “algebraically”?

A first answer

Implicative algebras are algebraic structures inducing triposes in such a way that every tripos can be seen as an implicative tripos.

Our answer

Arrow algebras generalize implicative algebras, perfectly factoring through the construction of realizability toposes from PCAs.

Arrow algebras

Arrow structures

An *arrow structure* is a complete poset $(A, \preccurlyeq)$ endowed with a binary operation $\rightarrow : A \times A \rightarrow A$ such that

$$\text{if } a' \preccurlyeq a \text{ and } b \preccurlyeq b', \text{ then } a \rightarrow b \preccurlyeq a' \rightarrow b'.$$

Elements of A should be thought of as *truth values*, or *pieces of evidence*. We refer to $\preccurlyeq$ as the *evidential* order.

Separator

A *separator* on an arrow structure $(A, \preccurlyeq, \rightarrow)$ is a subset $S \subseteq A$ such that:

1. if $a \in S$ and $a \preccurlyeq b$, then $b \in S$;
2. if $a \rightarrow b \in S$ and $a \in S$, then $b \in S$;
3. S contains the following *combinators*:
    - $\mathsf{k} \coloneqq \curlywedge_{a,b}\, a \rightarrow b \rightarrow a$;
    - $\mathsf{s} \coloneqq \curlywedge_{a,b,c} (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c)$;
    - $\mathsf{a} \coloneqq \curlywedge_{a,(b_i)_{i\in I},(c_i)_{i\in I}} \left(\curlywedge_{i\in I} a \rightarrow b_i \rightarrow c_i\right) \rightarrow a \rightarrow \left(\curlywedge_{i\in I} b_i \rightarrow c_i\right)$.

Arrow algebra

An *arrow algebra* $\mathcal{A}$ is a quadruple $(A, \preccurlyeq, \to, S)$ where $(A, \preccurlyeq, \to)$ is an arrow structure and S is a separator on it.

Examples

1. Implicative algebras.
2. Frames, with the separator $\{\top\}$.
3. The poset DA, for a PCA $\mathbb{A}$, with implication:

$$\alpha \to \beta := \downarrow\{\, c \in A \mid (\forall a \in \alpha)(ca\downarrow \text{ and } ca \in \beta \,\}$$

   and separator $\left\{\, \alpha \in DA \;\middle|\; \exists r \in \alpha \cap A^{\#} \,\right\}$.

Given an arrow algebra $\mathcal{A}$, the relation:

$$a \vdash b \iff a \to b \in S$$

is a preorder, and $(A, \vdash)$ carries the structure of a Heyting prealgebra. We refer to $\vdash$ as the *logical* order.

For any set I, pointwise order and implication define an arrow structure on the set A^I, which we can turn into an arrow algebra $\mathcal{A}^I$ with the *uniform power separator*:

$$\phi \in S^I \iff \bigcurlywedge_{i \in I} \phi(i) \in S$$

We denote with $\vdash_I$ the logical order in $\mathcal{A}^I$; explicitly:

$$\phi \vdash_I \psi \iff \bigcurlywedge_{i \in I} \phi(i) \to \psi(i) \in S$$

Theorem

For any arrow algebra $\mathcal{A}$, the functor:

$$P_{\mathcal{A}} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{HeytPre} \qquad \begin{array}{ccc} I & \longmapsto & (A^I, \vdash_I) \\ f \big\uparrow & & \big\downarrow -\circ f \\ J & \longmapsto & (A^J, \vdash_J) \end{array}$$

is a tripos having $\mathrm{id}_A \in P_{\mathcal{A}}(A)$ *as generic predicate.*

Notation

We denote with $\mathrm{AT}(\mathcal{A})$ the *arrow topos* $\mathrm{Set}[P_{\mathcal{A}}]$.

Implicative morphisms

Implicative morphisms underline the privileged role of the logical order and implication in arrow algebras.

Implicative morphism

Given two arrow algebras $\mathcal{A}$ and $\mathcal{B}$, an *implicative morphism* $f : \mathcal{A} \to \mathcal{B}$ is a function $f : A \to B$ satisfying:

1. $f(a) \in S_B$ for all $a \in S_A$;
2. $\curlywedge_{a,a' \in A} f(a \to a') \to f(a) \to f(a') \in S_B$;
3. for any subset $X \subseteq A \times A$,

$$\text{if} \quad \curlywedge_{(a,a') \in X} a \to a' \in S_A \quad \text{then} \quad \curlywedge_{(a,a') \in X} f(a) \to f(a') \in S_B.$$

Morphisms $\mathcal{A} \to \mathcal{B}$ can be ordered by restricting $\vdash_A$:

$$f \vdash f' \iff \bigcurlywedge_{a} f(a) \to f'(a) \in S_B$$

We denote with ArrAlg the preorder-enriched category of arrow algebras and implicative morphisms.

Lemma

Every implicative morphism is isomorphic to a monotone one.

Examples

1. Frame homomorphisms are implicative morphisms.
2. Morphisms of PCAs $D\mathbb{A} \to D\mathbb{B}$ are implicative morphisms.
3. For any partial applicative morphism of PCAs $f : \mathbb{A} \to \mathbb{B}$, the function:

$$\widetilde{f} : DA \to DB \qquad \widetilde{f}(\alpha) := \bigcup_{a \in \alpha} f(a)$$

is an implicative morphism $D\,\mathbb{A} \to D\,\mathbb{B}$. The assignment $f \mapsto \widetilde{f}$ defines a 2-functor $\mathrm{oPCA}_D \to \mathrm{ArrAlg}$ such that the map

$$\mathrm{oPCA}_D(\mathbb{A}, \mathbb{B}) \to \mathrm{ArrAlg}(D\,\mathbb{A}, D\,\mathbb{B})$$

(preserves and) reflects the order.

Transformations of arrow triposes

Implicative morphisms give rise to left exact transformations of arrow triposes, that is, (pseudo)natural transformations whose components are monotone maps preserving finite meets.

Proposition

For any implicative morphism $f : \mathcal{A} \to \mathcal{B}$, the map

$$\Phi_f^+ : P_{\mathcal{A}} \to P_{\mathcal{B}} \qquad (\Phi_f^+)_I(\phi) := f \circ \phi$$

is a left exact transformation of triposes. The assignment $f \mapsto \Phi_f^+$ *defines a 2-functor* $\mathsf{ArrAlg} \to \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})$*, where* $\mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})$ *is the preorder-enriched category of triposes and left exact transformations.*

More interestingly, the converse is also true.

Proposition

The 2-functor ArrAlg $\to$ $\mathrm{Trip}_{\mathrm{lex}}(\mathrm{Set})$ *is 2-fully faithful.*

Explicitly, this means that for any arrow algebras $\mathcal{A}$ *and* $\mathcal{B}$ *there is an equivalence of preorder categories:*

$$\mathrm{ArrAlg}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Trip}_{\mathrm{lex}}(\mathrm{Set})(P_{\mathcal{A}}, P_{\mathcal{B}})$$

Definition

A *geometric morphism* $Q \to P$ is a left exact transformation $\Phi : P \to Q$ having a *right adjoint* $\Psi : Q \to P$, meaning that $\Phi_I \dashv \Psi_I : Q(I) \to P(I)$ as maps of preorders for any set I.

In this perspective, it makes sense to try to characterize those implicative morphisms whose induced left exact transformation of arrow triposes has a right adjoint.

Computationally dense implicative morphism

An implicative morphism $f : \mathcal{A} \to \mathcal{B}$ is *computationally dense* if it admits a right adjoint in ArrAlg, that is, if there exists an implicative morphism $h : \mathcal{B} \to \mathcal{A}$ such that $fh \vdash \mathrm{id}_B$ and $\mathrm{id}_A \vdash fh$.

Examples

1. Frame homomorphisms coincide precisely with computationally dense implicative morphisms.
2. A partial applicative morphism of PCAs $f : \mathbb{A} \to \mathbb{B}$ is computationally dense if and only if so is $\widetilde{f} : D\,\mathbb{A} \to D\,\mathbb{B}$ as an implicative morphism. The assignment $f \mapsto \widetilde{f}$ defines a 2-functor $\mathsf{oPCA}_{D,\mathrm{cd}} \to \mathsf{ArrAlg}_{\mathrm{cd}}$ such that the map

$$\mathsf{oPCA}_{D,\mathrm{cd}}(\mathbb{A}, \mathbb{B}) \to \mathsf{ArrAlg}_{\mathrm{cd}}(D\,\mathbb{A}, D\,\mathbb{B})$$

is an equivalence of preorder categories.

The assignment $f \mapsto \Phi_f^+$ restricts to a 2-functor:

$$\mathrm{ArrAlg}_{\mathrm{cd}} \longrightarrow \mathrm{Trip}_{\mathrm{geo}}(\mathrm{Set})$$

where $\mathrm{Trip}_{\mathrm{geo}}(\mathrm{Set})$ is the preorder-enriched category of triposes and left exact transformations having a right adjoint between them.

A computationally dense $f : \mathcal{A} \to \mathcal{B}$ with right adjoint h induces the geometric morphism:

$$P_{\mathcal{A}} \underset{h\circ -}{\overset{f\circ -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} P_{\mathcal{B}}$$

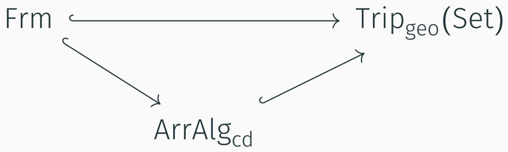
As in the previous case, the converse is also true.

Theorem

The 2-functor $\mathrm{ArrAlg}_{\mathrm{cd}} \to \mathrm{Trip}_{\mathrm{geo}}(\mathrm{Set})$ *is 2-fully faithful.*

Explicitly, this means that for any arrow algebras $\mathcal{A}$ *and* $\mathcal{B}$ *there is an equivalence of preorder categories:*

$$\mathrm{ArrAlg}_{\mathrm{cd}}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Trip}_{\mathrm{geo}}(\mathrm{Set})(P_{\mathcal{A}}, P_{\mathcal{B}})$$


Frm
Trip_geo(Set)
ArrAlg_cd


oPCA_D,cd
Trip_geo(Set)
ArrAlg_cd

Nucleus

Let $\mathcal{A}$ be an arrow algebra. A *nucleus* j on $\mathcal{A}$ is a function $j : A \to A$ such that:

1. if $a \preccurlyeq b$ then $ja \preccurlyeq jb$;
2. $\curlywedge_{a,b \in A} j(a \to b) \to ja \to jb \in S$;
3. $\curlywedge_{a \in A} a \to ja \in S$;
4. $\curlywedge_{a \in A} jja \to ja \in S$.

Example

Nuclei on a frame are precisely nuclei in the localic sense.

Every nucleus j on $\mathcal{A} = (A, \preccurlyeq, \to, S)$ determines an arrow algebra $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$ over the same poset, but with a new implication and separator:

$$a \to_j b := a \to jb \qquad S_j := \{\, a \in A \mid ja \in S \,\}$$

Lemma

id_A is a computationally dense implicative morphism $\mathcal{A} \to \mathcal{A}_j$, with right adjoint j.

Theorem

For any arrow algebra $\mathcal{A}$, the construction of the arrow tripos yields an equivalence of preorder categories:

$$\mathsf{N}(\mathcal{A})^{\mathsf{op}} \simeq \mathsf{SubTrip}(P_{\mathcal{A}})$$

Moreover, every subtripos of $P_{\mathcal{A}}$ is up to equivalence of the form:

$$P_{\mathcal{A}_j} \underset{j \circ -}{\overset{\mathbf{id}_A \circ -}{\underset{\bot}{\rightleftarrows}}} P_{\mathcal{A}}$$

for some $j \in \mathsf{N}(\mathcal{A})$.

Thank you!