# Categorical Structure in Theory of Arithmetic

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# Introduction

Let $\mathbb{T}$ be some sufficiently strong theory of arithmetic. A formula is $\Sigma_1$ if it is provably equivalent to a coherent formula ($\top, \wedge, \bot, \vee, \exists$).

**Proposition**

A subset of $\mathbb{N}$ is r.e. iff it is definable by a $\Sigma_1$-formula.

A function $f : \mathbb{N}^k \to \mathbb{N}$ is *provably total (recursive)* in $\mathbb{T}$ if:

- There is a $\Sigma_1$-formula $\varphi_f(\overline{x}, y)$ defining the graph of $f$;
- $\mathbb{T} \vdash \forall \overline{x} \exists_! y \varphi_f(\overline{x}, y)$.

This class will be denoted as $\mathfrak{R}(\mathbb{T})$. It measures the strength of $\mathbb{T}$.

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# Complexity and Arithmetic — Cont.

Logicians have considered a wide variety of arithmetic theories,

- PA, $I\Sigma_n$, EA, $\mathrm{PA}^-$, $Q$, $S^k_n$, ...

When $\mathbb{T}$ is $I\Sigma_1$ (PA but with induction restricted to $\Sigma_1$-formulas):

**Theorem (⋆)**

*Provably total functions in $I\Sigma_1$ are exactly p.r. functions.*

+ Another equivalent way of characterising p.r. functions.
+ $\mathfrak{R}(\mathbb{T})$ is intimately related to the proof-theoretic ordinal of $\mathbb{T}$.
- Most/All proofs are like "programs on machine code".

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## Categorical Logic

*Coherent* logic is the fragment of first-order logic with:

- Formulas built up from $\top, \wedge, \bot, \vee, \exists$;
- Proofs formulated in sequent style $\varphi \vdash_{\bar{x}} \psi$;

# Categorical Logic — Cont.

Any $\mathbb{T}$ has a *syntactic category* $\mathcal{C}[\mathbb{T}]$ encapsulating itself:

- Objects are formulas (with contexts) in $\mathbb{T}$ / $\sim_\alpha$;
- Morphisms $\theta : \varphi(\bar{x}) \to \psi(\bar{y})$ are $\mathbb{T}$-functional formulas / $\sim_{\mathbb{T}}$:

$$\theta(\bar{x}, \bar{y}) \vdash_{\bar{x},\bar{y}} \varphi(\bar{x}) \wedge \psi(\bar{y})$$

$$\varphi(\bar{x}) \vdash_{\bar{x}} \exists \bar{y} \theta(\bar{x}, \bar{y})$$

$$\theta(\bar{x}, \bar{y}) \wedge \theta(\bar{x}, \bar{z}) \vdash_{\bar{x},\bar{y},\bar{z}} \bar{y} = \bar{z}$$

**Functorial Semantics**

Sending a model $M$ to a functor $\varphi \mapsto [\![\varphi]\!]_M$ gives an equivalence

$$\mathbf{Coh}(\mathcal{C}[\mathbb{T}], \mathbf{Set}) \simeq \mathrm{Mod}(\mathbb{T}).$$

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We want to find a suitable *coherent* theory of arithmetic $\mathbb{T}$ that faithfully represents the relevant fragment of $I\Sigma_1$:

**Theorem (Correctness)**

*The interpretation of $\mathbb{T}$ into $I\Sigma_1$ induces an equivalence*

$$\mathcal{C}[\mathbb{T}] \simeq \mathcal{C}[I\Sigma_1]_{\Sigma_1},$$

*where $\mathcal{C}[I\Sigma_1]_{\Sigma_1}$ is the full subcategory of $\Sigma_1$-formulas.*

Given such $\mathbb{T}$, the *subject* of $(\star)$ can be easily recognised in $\mathcal{C}[\mathbb{T}]$:

- Let $[n]$ denote $\bigwedge_{1\leq i\leq n} x_i = x_i$. We think of $[1]$ as the natural numbers in $\mathcal{C}[\mathbb{T}]$, with $[n] \cong [1]^n$ in $\mathcal{C}[\mathbb{T}]$.

**Observation**

$\mathcal{C}[\mathbb{T}]([n],[1])$ corresponds to provably total functions of $\mathbb{T}$ ($I\Sigma_1$).

# A Coherent Theory of Arithmetic — Cont.

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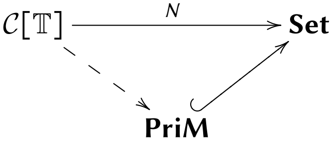
# Strategy

According to categorical logic, the standard model $\mathbb{N}$ induces:

$$\mathcal{C}[\mathbb{T}] \xrightarrow{N} \mathbf{Set}$$

$N$ maps every $\theta : [n] \to [1]$ to the function $\mathbb{N}^n \to \mathbb{N}$ it defines. The hard part of $(\star)$ is to show the images of these morphisms are p.r.

($\star$) now is equivalent to the existence of a factorisation:




where **PriM** morally is a category with

- Objects being r.e. subsets of $\mathbb{N}^n$;
- Morphisms being p.r. functions.

## Strategy — Cont.

Such a situation begs for *initiality* result: $\mathcal{C}[\mathbb{T}]$ should be initial among certain class of categories containing **PriM** and **Set**.

**Theorem (Initiality)**

$\mathcal{C}[\mathbb{T}]$ *is initial among coherent categories with a parametrised natural numbers object (PNO).*

Examples of coherent categories with a PNO:

- **Set**, **PriM**, any topos with a natural numbers object ...

Now $(\star)$ is implied by **Correctness** + **Initiality**.

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# Coherent Theory of Arithmetic

The design of $\mathbb{T}$ should take into account the following points:

- **Validity**: What's present in $\mathbb{T}$ should be universally valid in all coherent categories with PNO, and preserved by such functors.
- **Strength**: $\mathbb{T}$ should be strong enough for $\mathcal{C}[\mathbb{T}]$ to have a PNO.

**Validity** + **Strength** = **Initiality**.

# Towards a Coherent Theory of Arithmetic

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# Construction of Coherent Arithmetic

We construct $\mathbb{T}$ as follows:

- It has a constant 0.
- It has all primitive function names *PR* as function symbols, plus their corresponding defining axioms.
- Besides coherent logic, it has an induction rule:

$$\frac{\varphi(\overline{x}) \vdash_{\overline{x}} \psi(\overline{x}, 0) \qquad \varphi(\overline{x}) \wedge \psi(\overline{x}, y) \vdash_{\overline{x}, y} \psi(\overline{x}, sy)}{\varphi(\overline{x}) \vdash_{\overline{x}, y} \psi(\overline{x}, y)}$$

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# Proof of Initiality

# Parametrised Natural Number Object

In a Cartesian category $\mathcal{C}$, an object $N$ is a PNO if we have

$$1 \xrightarrow{0} N \xleftarrow{s} N$$

such that for any $g : A \to B$ and $h : A \times N \times B \to B$, there is a *unique* map $rec_{g,h} : A \times N \to B$ making the following commute,

$$\begin{array}{ccccc} A & \xrightarrow{\langle \mathsf{id},0\rangle} & A \times N & \xleftarrow{\mathsf{id} \times s} & A \times N \\ & \searrow^{g} & \downarrow{\scriptstyle rec_{g,h}} & & \downarrow{\scriptstyle \langle \mathsf{id}, rec_{g,h}\rangle} \\ & & B & \xleftarrow[h]{} & A \times N \times B \end{array}$$

**Theorem**

*For a PNO $N$ in $\mathcal{C}$, there is a unique map $ev : PR \to Mor(\mathcal{C})$, which is preserved by Cartesian functors preserving the PNO.*

**Proof.**

Consider the following diagramme:




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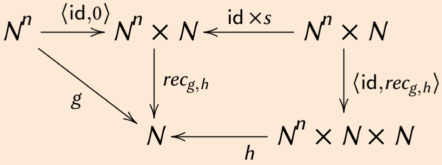


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# Induction Principle of PNO

**Theorem**

*The induction rule is valid for a PNO: For any object X, if*

$$X \vDash \varphi(x) \vdash \psi(x, 0) \quad X \times N \vDash \varphi(x) \wedge \psi(x, n) \vdash \psi(x, sn),$$

*then we also have*

$$X \times N \vDash \varphi(x) \vdash \psi(x, n).$$

**Proof.**

Take the usual proof of induction of an NNO to the parametrised version. □

Together they have shown **Validity**.

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# PNO in $\mathcal{C}[\mathbb{T}]$

The remaining work is to show $[1] =: N$ is a PNO in $\mathcal{C}[\mathbb{T}]$:

$$\begin{array}{ccccc} \varphi & \xrightarrow{\langle \mathrm{id},0\rangle} & \varphi \times N & \xleftarrow{\mathrm{id}\times s} & \varphi \times N \\ & \searrow^{\gamma} & \Big\downarrow{\scriptstyle \mathit{rec}_{\gamma,\theta}} & & \Big\downarrow{\scriptstyle \mathit{rec}_{\gamma,\theta}} \\ & & \psi & \xleftarrow[\theta]{} & \psi \end{array}$$

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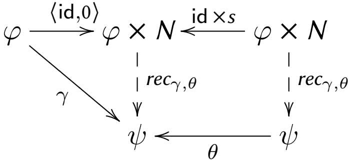


This requires us to show we can encode finite lists of numbers in $\mathbb{T}$:

$$rec_{\gamma,\theta}(x, n, y) := \exists l(|l| = sn \wedge \gamma(x, l_0) \wedge \forall u{<}n\, \theta(l_u, l_{su}) \wedge l_n = y).$$

This is standard in arithmetic.

## Remark on Correctness

To conclude $(\star)$ then, we only need to show **Correctness**:

- It is a classical result in topos theory that classical logic is *conservative* over the coherent fragment.
- We can also use pure proof theory techniques to show this: cut-elimination/normalisation.

**Conclusion**: $(\star)$ is true by the *structural* reason that the $\Sigma_1$-fragment of $I\Sigma_1$ presents the initial coherent category with PNO.

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## The End

Thanks for Listening!

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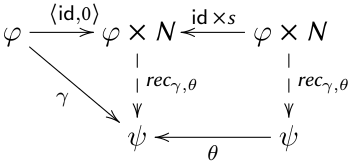


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This is standard in meta-logic practice.

## The Lie I've been Telling

$\Sigma_1$-formulas of $I\Sigma_1$ also allow *bounded* universal quantification:

- For the above construction to work, we also requires bounded universal quantifiers in $\mathbb{T}$, and the actual $\mathbb{T}$ has them.
- For our proof to work, we further need to show **Validity** for them. This can be done in a *cohernet* setting.
- Using this, we can show **Strength**, and conclude **Initiality**.