Models for synthetic calculus

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1 Abstract

We will look at a rigorous approach for considering infinitesimals: quantities that are infinitely small. The theory that we will consider is *synthetic calculus*, which calls for a *constructive approach* to mathematics. Because this may lead to some discomfort, we will actually look at *models* for synthetic calculus, which means that we are going to construct the infinitesimals as actual, classical objects. This way, our thinking can remain completely classical, and we will see how the constructive logic arises from an external, classical point of view.

2 Loci

By $C^\infty(\mathbb{R}^d)$ we denote the (commutative) ring of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}$, with the operations defined pointwise.

An ideal I in the ring $C^\infty(\mathbb{R}^d)$ is an additive subgroup satisfying that for every $f \in I$ and every $g \in C^\infty(\mathbb{R}^d)$, $f \cdot g \in I$. The ideal generated by an arbitrary subset $J \subset C^\infty(\mathbb{R}^d)$ is the smallest ideal containing J. We write (g) for the ideal generated by a function $g \in C^\infty(\mathbb{R}^d)$.

Given an ideal $I \subset C^\infty(\mathbb{R}^d)$ we can consider the quotient ring $C^\infty(\mathbb{R}^d)/I$, where the operations are defined on the representatives.

Every ring of the form $C^\infty(\mathbb{R}^d)/I$ induces a so-called *locus*, which is just the ring itself, except that we now write it as $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$.

A *morphism* from a locus $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ to a locus $\ell(C^\infty(\mathbb{R}^m)/J)$ is an equivalence class $[\phi]$ of smooth maps $\phi : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\forall f \in J, f \circ \phi \in I.$$

where the equivalence relation is defined by $\phi \sim \psi$ if for $j = 1, \ldots, m$,

$$\phi_j - \psi_j \in I.$$

We will sometimes denote the set of morphisms from a locus $\ell(A)$ to a locus $\ell(B)$ by

$$\mathsf{Hom}(\ell(A), \ell(B)).$$

Proposition 2.1. Given two morphisms

$$[\psi] : \ell\Big(C^\infty(\mathbb{R}^d)/I\Big) \to \ell(C^\infty(\mathbb{R}^m)/J)$$
$$[\phi] : \ell(C^\infty(\mathbb{R}^m)/J) \to \ell(C^\infty(\mathbb{R}^n)/K)$$

the equivalence class $[\phi \circ \psi]$ is itself a morphism from the locus $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ to the locus $\ell(C^\infty(\mathbb{R}^n)/K)$. Moreover, this equivalence class is independent of the particular representatives ψ and ϕ, and is called the composition of $[\phi]$ and $[\psi]$, and is denoted by $[\phi] \circ [\psi]$.

Proof. First note that for all $f \in C^\infty(\mathbb{R}^n)$ such that $f \in K$, we have that

$$f \circ \phi \in J$$

and therefore

$$f \circ \phi \circ \psi \in I$$

hence $[\phi \circ \psi]$ is a morphism.

Assume $\phi \sim \tilde{\phi}$ and $\psi \sim \tilde{\psi}$. We need to show that $[\phi \circ \psi] = [\tilde{\phi} \circ \tilde{\psi}]$. It suffices to show that $[\phi \circ \psi] = [\tilde{\phi} \circ \psi]$ and $[\phi \circ \psi] = [\phi \circ \tilde{\psi}]$.

Let $j \in \{1, \ldots, n\}$. Note that

$$(\phi \circ \psi)_j - (\tilde{\phi} \circ \psi)_j = (\phi_j - \tilde{\phi}_j) \circ \psi \in I$$

because $[\psi]$ is a morphism and $\phi_j - \tilde{\phi}_j \in J$.

By Hadamard's lemma, there exist smooth functions $q_1, \ldots, q_m : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}^m$,

$$\phi_j(x) - \phi_j(y) = \sum_{i=1}^{n} (x_i - y_i) q_i(x, y).$$

Therefore, for all $x \in \mathbb{R}^d$

$$\phi_j(\psi(x)) - \phi_j(\tilde{\psi}(x)) = \sum_{i=1}^{m} (\psi_i(x) - \tilde{\psi}_i(x)) q_i(\psi(x), \tilde{\psi}(x)).$$

It follows that $\phi_j \circ \psi - \phi_j \circ \tilde{\psi} \in I$. □

These notes try to not rely on category theory, but for readers familiar it may be useful to once in a while connect to these concepts. A category consists of a collection of objects, for each pair of objects a collection of so-called morphisms, for every object an identity morphism of the object to itself and an associative composition operation for morhpisms.

Definition 2.2 (The category of loci $\mathbb{L}$)**.** The objects in the category of loci $\mathbb{L}$ are quotient rings of the form $C^\infty(\mathbb{R}^d)/I$ for some $d \in \mathbb{N}_0$ and an ideal $I \subset C^\infty(\mathbb{R}^d)$. Such an object is denoted by $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$.

Morphisms in this category from an object $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ to an object $\ell(C^\infty(\mathbb{R}^m)/J)$ are equivalence classes $[\phi]$ of smooth maps $\phi : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\forall f \in J, f \circ \phi \in I$$

where the equivalence relation is defined by $\phi \sim \psi$ if for $j = 1, \ldots, m$,

$$\phi_j - \psi_j \in I.$$

3 Dictionary of spaces

We present a small dictionary of some spaces in synthetic calculus. Some are analogs of classical spaces, like the real numbers, and others are new, such as the first-order infinitesimals.

Although I tried to write these notes so you don't need to know what presheaves are, it is nonetheless going to be useful to start using the word *presheaves* for the objects that we construct.

As a bit of motivation, it is useful to embed the category of loci into a larger category (the category of presheaves) because some constructions of presheaves are possible that are not always possible for loci: although some function spaces made from loci are again loci, in general they are just presheaves. A presheaf is called representable if it is the Yoneda embedding of a locus, so a different way of saying the last sentence is that the presheaf of functions from one representable presheaf to another representable presheaf is not necessarily representable.

The embedding from the category of loci to the category of presheaves is called the Yoneda embedding. The Yoneda embedding assigns to a locus $\ell(A)$ the functor

$$\mathbb{Y}(\ell(A)) := \mathsf{Hom}(-, \ell(A))$$

which is the functor of morphisms to $\ell(A)$. By the Yoneda lemma, the Yoneda embedding $\mathbb{Y}$ is a fully faithful embedding, which basically means that $\mathbb{L}$ is just a subcategory of the category of presheaves on $\mathbb{L}$.

Here is the dictionary:

i. The smooth line (or the reals):

$$R := \mathbb{Y}(\ell(C^{\infty}(\mathbb{R}))) := \mathsf{Hom}(-, \ell(C^{\infty}(\mathbb{R}))).$$

ii. The point:

$$1 := \mathbb{Y}(\ell(C^{\infty}(\mathbb{R})/(x))).$$

iii. The first-order infinitesimals:

$$D := \mathbb{Y}(\ell\Big(C^{\infty}(\mathbb{R})/(x^2)\Big)).$$

iv. The (strictly) positive reals:

$$R_{>0} := \mathbb{Y}\left(\ell\Big(C^\infty(\mathbb{R}^2)/(y\mathbf{1}_{>0}(x)-1)\Big)\right)$$

where $\mathbf{1}_A$ stands for the characteristic function of a set A.

v. The nonpositive reals:

$$R_{\leq 0} := \mathbb{Y}\left(\ell\big(C^\infty(\mathbb{R})/m^\infty_{\leq 0}\big)\right)$$

where $m^\infty_{\leq 0}$ of functions that are flat on $(-\infty, 0]$.

4 Analogs of set-theoretic concepts

We first define elements of general presheaves, but you are invited to skip this definition and just read what elements of representable presheaves are.

Definition 4.1 (Element)**.** A *(restricted generalized) element* of a presheaf F is a morphism (i.e. a natural transformation) from some presheaf $\mathbb{Y}(\ell\big(C^\infty(\mathbb{R}^d)/I\big))$ to F. Here $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ is called the *stage* of the element. We will also use the terminology *an element of F* at stage $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ as an element with domain $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$.

The next proposition characterizes elements of representable presheaves.

Proposition 4.2 (Element of a (Yoneda embedding of) a locus)**.** A *(restricted generalized) element* of a representable presheaf $\mathbb{Y}(\ell(C^\infty(R^m)/J))$ corresponds to a morphism

$$\ell\Big(C^\infty(\mathbb{R}^d)/I\Big) \to \ell(C^\infty(R^m)/J)\,.$$

Here $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ is called the *stage* of the element.

Definition 4.3 (Restriction of an element)**.** Let ξ be an element of F at stage $\ell(A)$, and let $\alpha : \ell(B) \to \ell(A)$, then *the restriction of ξ along α*, denoted by $\xi|\alpha$, is the element of F at stage $\ell(B)$ given by $\xi \circ \alpha$.

Definition 4.4 (Product)**.** The product of two loci $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ and $\ell(C^\infty(\mathbb{R}^m)/J)$ is given by $\ell\big(C^\infty(\mathbb{R}^{d+m})/(I,J)\big)$, where (I,J) is the ideal $(I \circ \pi_1 + J \circ \pi_2)$ where $\pi_1 : \mathbb{R}^{d+m} \to \mathbb{R}^d$ and $\pi_2 : \mathbb{R}^{d+m} \to \mathbb{R}^m$ are the projection maps.

The projections of $\ell\big(C^\infty(\mathbb{R}^{d+m})/(I,J)\big)$ to the first and second components are given by $[\pi_1]$ and $[\pi_2]$ respectively.

Definition 4.5. Let $\ell(A) = \ell\big(C^\infty(\mathbb{R}^d)/I\big)$, $\ell(B) = \ell(C^\infty(\mathbb{R}^m)/J)$ and $\ell(C) = \ell(C^\infty(\mathbb{R}^n)/K)$. Then an element of the function space

$$\mathbb{Y}(B)^{\mathbb{Y}(C)}$$

at stage $\ell(A)$ is a morphism

$$\ell(A) \times \ell(C) \to \ell(B)\,.$$

The evaluation of a morphism f of $\mathbb{Y}(B)^{\mathbb{Y}(C)}$ at stage $\ell(A)$ in an element c of $\mathbb{Y}(\ell(C))$ at stage $\ell(A)$, is given by

$$f \circ (\mathrm{id}_{\ell(A)}, c).$$

The restriction of a morphism f of $\mathbb{Y}(B)^{\mathbb{Y}(C)}$ at stage $\ell(A)$ along $\alpha : \ell(E) \to \ell(A)$ is defined by

$$f|\alpha := f \circ (\alpha, \mathrm{id}_{\ell(C)}).$$

5 Interpretation of the logical language

We will now try to explain how to read logical statements. As a first example, if we have a pair of elements a, b of R at stage $\ell(A)$, we could make the statement that

$$a \cdot b = b \cdot a$$

Moreover, this statement is valid for these elements, and we express this by saying that $a \cdot b = b \cdot a$ is satisfied at stage $\ell(A)$. We write this as $\ell(A) \Vdash a \cdot b = b \cdot a$.

Below, the variables $F_1, F_2, \ldots$ denote presheaves.

i. Often if we have a collection of variables at a given stage $\ell(A)$, say $a_1, \ldots, a_d$, then we know what it means for some formula $S = \phi(a_1, \ldots, a_d)$ to hold. We call this: the statement S is satisfied at stage $\ell(A)$ and write this as $\ell(A) \Vdash S$. The relation $\ell(A) \Vdash S$ is really defined inductively, as we try to describe below.

ii. We say *a statement S holds* if for every $\ell(A) \in \mathbb{L}$, $\ell(A) \Vdash S$.

iii. The notation (or rather the subfunctor of $F_1 \times \cdots \times F_n$)

$$\{(x_1, \ldots, x_n) \in F_1 \times \cdots \times F_n \mid \varphi(x_1, \ldots, x_n)\}$$

is defined through:

for all $\ell(A) \in \mathbb{L}$ and all elements a_1 of F_1, $\ldots$, a_n of F_n all at stage $\ell(A)$,

$$\Big[(a_1, \ldots, a_n) \in \{(x_1, \ldots, x_n) \in F_1 \times \cdots \times F_n \mid \varphi(x_1, \ldots, x_n)\}$$

at stage $\ell(A)$ if and only if

$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n)\Big]$$

iv.

$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n) \wedge \psi(a_1, \ldots, a_n)$$

if and only if both

$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n)$$

and

$$\ell(A) \Vdash \psi(a_1, \ldots, a_n)$$

v.
$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n) \vee \psi(a_1, \ldots, a_n)$$

if either
$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n)$$

or
$$\ell(A) \Vdash \psi(a_1, \ldots, a_n)$$

vi.
$$\ell(A) \Vdash \exists x \in F, \varphi(x, a_1, \cdots, a_n)$$

if and only if there exists an $a : \ell(A) \to F$ such that
$$\ell(A) \Vdash \varphi(a, a_1, \ldots, a_n)$$

vii.
$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n) \implies \psi(a_1, \cdots, a_n)$$

if and only if for every locus $\ell(B)$ and every $f : \ell(B) \to \ell(A)$ in $\mathbb{L}$, if
$$\ell(B) \Vdash \varphi(a_1|f, \ldots, a_n|f)$$

then
$$\ell(B) \Vdash \psi(a_1|f, \ldots, a_n|f).$$

viii.
$$\ell(A) \Vdash \forall x \in F, \varphi(x, a_1, \cdots, a_n)$$

if and only if for every locus $\ell(B)$ and every $f : \ell(B) \to \ell(A)$ in $\mathbb{L}$ and every $b : \ell(B) \to F$,
$$\ell(B) \Vdash \varphi(b, a_1|f, \ldots, a_n|f)$$

Theorem 5.1 (Functoriality of $\Vdash$)**.** If
$$\ell(A) \Vdash \varphi(a_1, \ldots, a_n)$$

and $\alpha : \ell(E) \to \ell(A)$ is a morphism, then
$$\ell(E) \Vdash \varphi(a_1|\alpha, \ldots, a_n|\alpha)$$

6 Exercises

Exercise 6.1. Prove that a real (i.e. an element of R) at stage $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ is an equivalence class $\phi \mod I$.

Exercise 6.2. Prove that an element of the point (i.e. an element of the representable presheaf 1) at stage $\ell(C^\infty(\mathbb{R}^n)/I)$ is the equivalence class $(0 \mod I)$.

Exercise 6.3. Prove that a first-order infinitesimal at stage $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ is an equivalence class $\phi \mod I$ with $\varphi^2 \in I$.

Exercise 6.4. Prove that $D = \{x \in R \mid x^2 = 0\}$.

Exercise 6.5. Try to prove that for every $x \in D$, $x = 0$. What goes wrong?

Exercise 6.6. Try to prove that there exists an element $x \in D$ such that $x \neq 0$. What goes wrong?

Exercise 6.7. Prove that

$$\forall x, y \in R, x \cdot y = y \cdot x,$$

i.e., prove that for every stage $\ell(A) \in \mathbb{L}$ and every pair of elements $a, b \in R$ at stage $\ell(A)$,

$$a \cdot b = b \cdot a.$$

Exercise 6.8. Prove that an element of R^R at stage $\ell(C^\infty(\mathbb{R}^m)/I)$ is an equivalence class $F \mod (I,(0))$ with $F : \mathbb{R}^{m+1} \to \mathbb{R}$ smooth.

Exercise 6.9. Prove that an element of R^D at stage $\ell\big(C^\infty(\mathbb{R}^d)/I\big)$ is an equivalence class $\phi \mod (I,(x^2))$ where $\phi : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth map.

In the next two exercises, we will prove the so-called Kock-Lawvere axiom, namely

$$\forall f \in R^D, \exists!(a,b) \in R \times R, \forall x \in D, f(x) = a + b \cdot x.$$

Exercise 6.10. Prove that

$$\forall f \in R^D, \exists (a,b) \in R \times R, \forall x \in D, f(x) = a + b \cdot x.$$

i. Verify that we need to show that for every locus $\ell(A)$ and every element f of R^D at stage $\ell(A)$,

$$\ell(A) \Vdash \exists (a,b) \in R \times R, \forall x \in D, (f(x) = a + b \cdot x)$$

ii. Let $\ell(A) = \ell\big(C^\infty(\mathbb{R}^d)/I\big)$ be an arbitrary locus. Verify that it suffices to show that there exists an element $(a,b) \in R \times R$ at stage $\ell(A)$ such that

$$\forall x \in D, f(x) = a + b \cdot x.$$

iii. Make a suitable choice for (a,b) at stage $\ell(A)$ (or delay this...)

iv. Verify that it suffices to show that for every locus $\ell(E)$ and every $\alpha : \ell(E) \to \ell(A)$ and every $c : \ell(E) \to D$,

$$\ell(E) \Vdash (f|\alpha)(c) = (a|\alpha) + (b|\alpha) \cdot c.$$

v. Let $\ell(E)$ be an arbitrary locus and let $\alpha : \ell(E) \to \ell(A)$ and $c : \ell(E) \to D$. Verify that it suffices to show that

$$\ell(E) \Vdash f \circ (\alpha, \mathrm{id}_D) \circ (\mathrm{id}_{\ell(E)}, c) = (a \circ \alpha) + (b \circ \alpha) \cdot c.$$

vi. Verify that it suffices to show that

$$\ell(A) \times D \Vdash f \circ (p_1, \mathrm{id}_D) \circ (\mathrm{id}_{\ell(E)}, p_2) = (a \circ p_1) + (b \circ p_1) \cdot p_2,$$

where p_1 and p_2 are the projections to $\ell(A)$ and D respectively.

vii. Prove that

$$\ell(A) \times D \Vdash f \circ (p_1, \mathrm{id}_D) \circ (\mathrm{id}_{\ell(E)}, p_2) = (a \circ p_1) + (b \circ p_1) \cdot p_2,$$

where p_1 and p_2 are the projections to $\ell(A)$ and D respectively.

Exercise 6.11. Prove that

$$\forall (a,b,u,v) \in R^4, (\forall x \in D, a + b \cdot x = u + v \cdot x) \implies (a = u \wedge b = v).$$