Algebraic K-Theory of Persistence Modules

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February 2, 2024

algo

Algorithms TU Eindhoven

Outline

1. Persistence modules
2. Algebraic K-theory of persistence modules

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Persistence Modules

A **persistence module** is a functor

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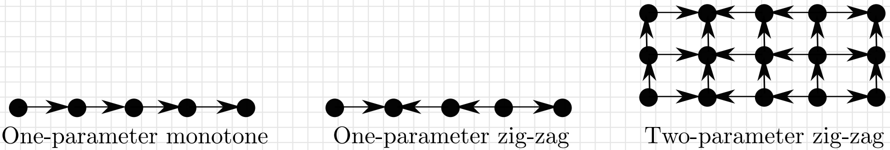
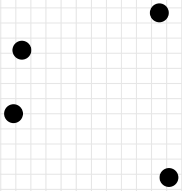


Modules From Filtrations



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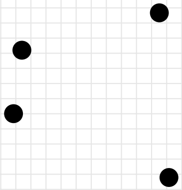

$$\text{radius} = 0$$

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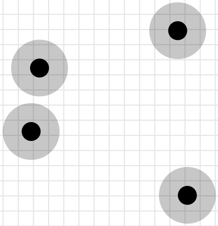

$$\text{radius} = 10$$

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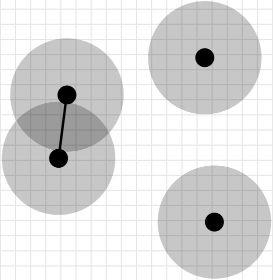

$$\text{radius} = 14$$

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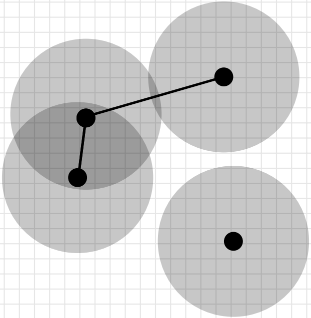

$$\text{radius} = 25$$

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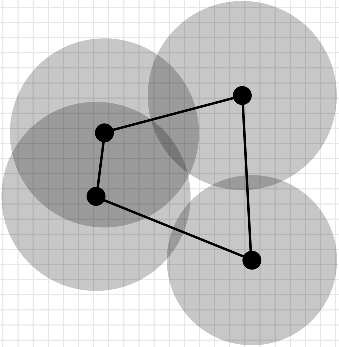

$$\text{radius} = 37$$

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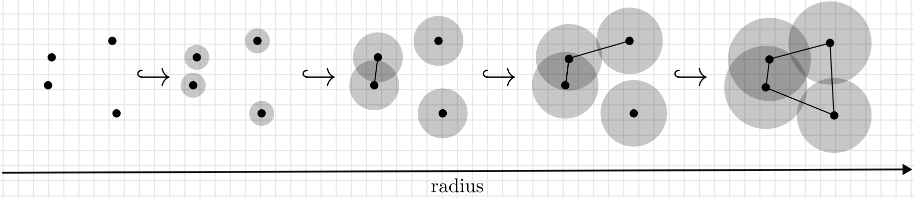

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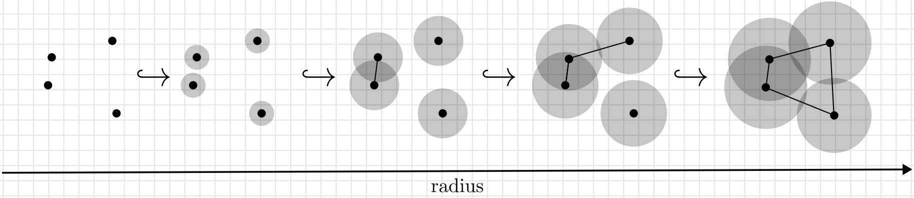

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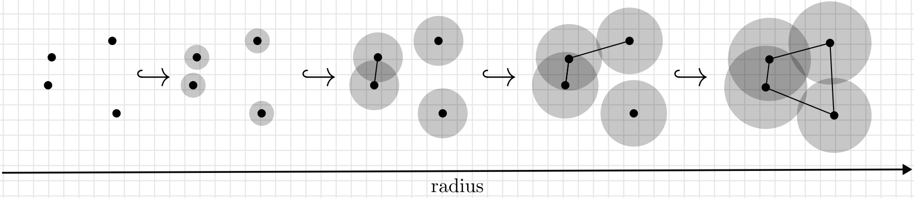

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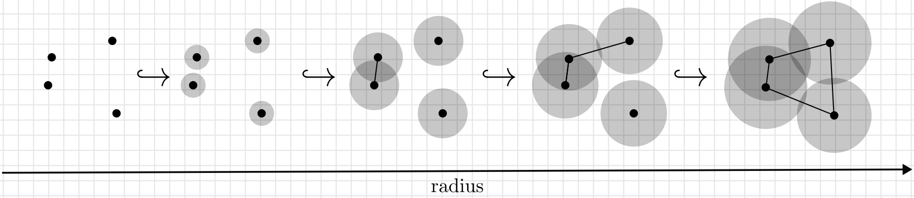

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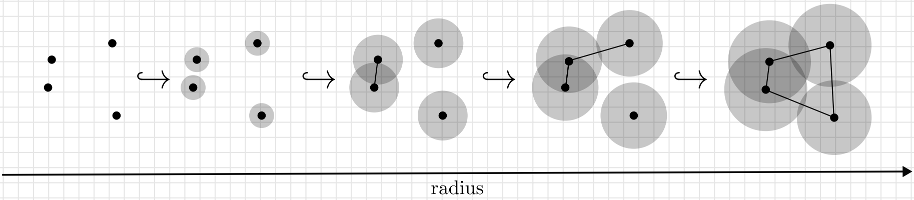

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- Notice there's a related functor $\mathbb{P} : \mathbb{R} \to \text{Vect}$.
- Using additional filter parameters is the usual way to get higher-dimensional persistence modules.

Zig-Zag Grid Modules

- We focus on **d-parameter zig-zag grid persistence modules**, where $I = I_1 \times I_2 \times \ldots \times I_d$, where each I_n is nontrivial and finite. We consider the objects as subsets of $\mathbb{R}$ (but possibly with different ordering).

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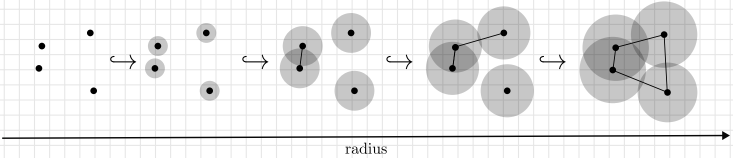

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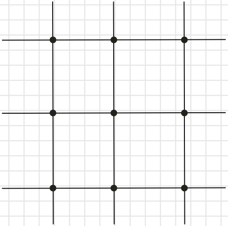
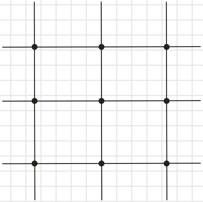


Combinatorial Entrance Path Category

I cubulates $\mathbb{R}^d$, and we stratify $\mathbb{R}^d$ by declaring each i cube of the cubulation to be an i-strata. Call this stratified space $(\mathbb{R}^d; I)$ (a **cubical manifold**).



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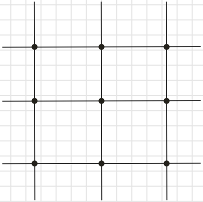

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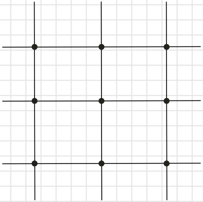

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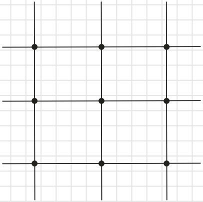

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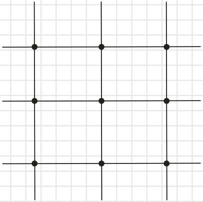

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Use the notation $\mathrm{pMod}^A(X) := \mathrm{Fun}(\mathrm{Ent}_\Delta(\mathrm{X}; \mathrm{I}), \mathrm{A})$.

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- Here, we forget parameter values and only keep track of the poset structure of I.

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- K-groups can provide a rich set of invariants of the input category.
- E.g., in the case of persistence modules, K_0 is the natural home for invariants like Euler characteristic curves, K_1 contains information about transformations between persistence modules, etc.

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Suppose that $A \to B \to C$ is a standard split short exact sequence of Waldhausen categories. Then, there is the following equivalence of spectra:

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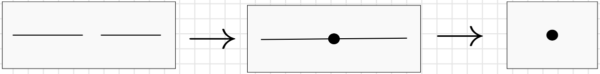

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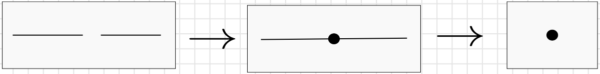

This splits! (and is standard) So ...

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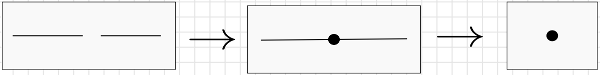

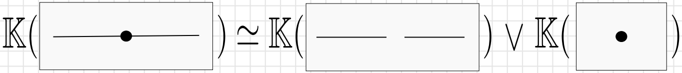

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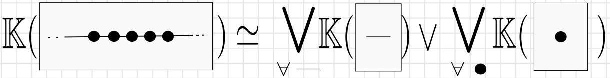

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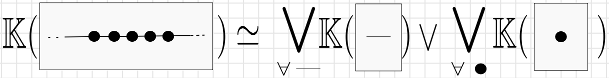

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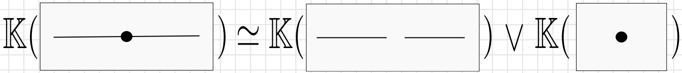

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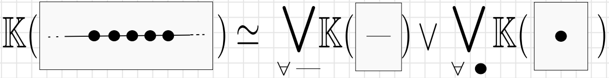

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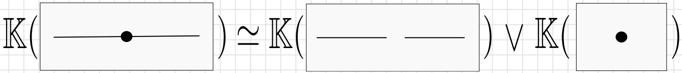

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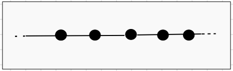

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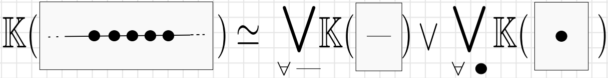

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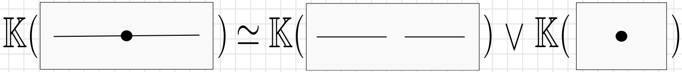

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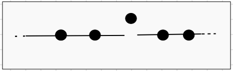

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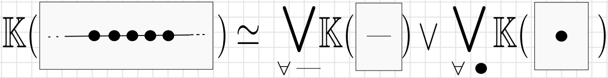

$\mathbb{K}$-theory of Zig-Zag Persistence Modules

K-Theory of One-Parameter Persistence Modules (Grady, S.)

Let X be a cubical one-manifold with finite zero-strata. Then there is an equivalence of spectra

$$\mathbb{K}(\mathrm{pMod}(X)) \simeq \bigvee_{x_1 \in X_1} \mathbb{K}(\mathrm{pMod}(x_1)) \vee \bigvee_{x_0 \in X_0} \mathbb{K}(\mathrm{pMod}(x_0)),$$

where X_i is the set of i-strata of X.

Extension to Multi-Parameter Modules

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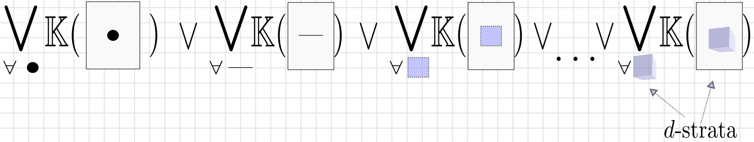
Additivity For Closed Sub-Stratified Spaces (Grady, S.)

Let X be a cubical manifold and let B denote a closed sub-stratified space of X. Then there is an equivalence of spectra

$$\mathbb{K}(\mathrm{pMod}(X)) \simeq \mathbb{K}(\mathrm{pMod}(X \setminus B)) \vee \mathbb{K}(\mathrm{pMod}(B)).$$

Additivity Over Strata

CLAIM: Let X be a cubical manifold that can be tightly embedded as a substratified manifold of some $(\mathbb{R}^d; I)$. The K-theory of persistence modules over X is equivalent with



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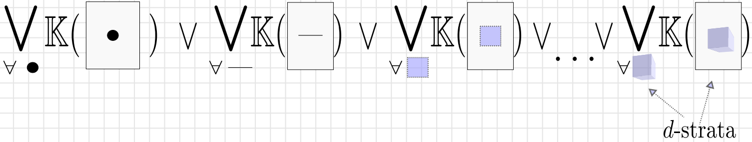
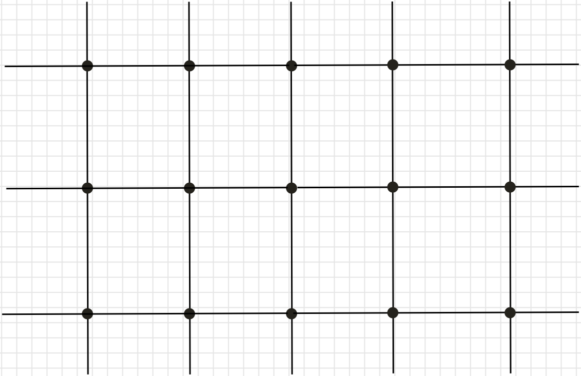


To extend to multi-parameter persistence modules, induct on d, the number of parameters, and on $h = \max_{n=1}^{d}\{|I_n|\}$ the *height* of the module[1].

[1]where d minimal such that X embeds as a stratified space into $(\mathbb{R}^d; I)$

Induction on Height

For the height induction, start with a cubical manifold with height h, then try to break it into pieces with height less than h (and each piece we remove needs to be closed in the space containing it).



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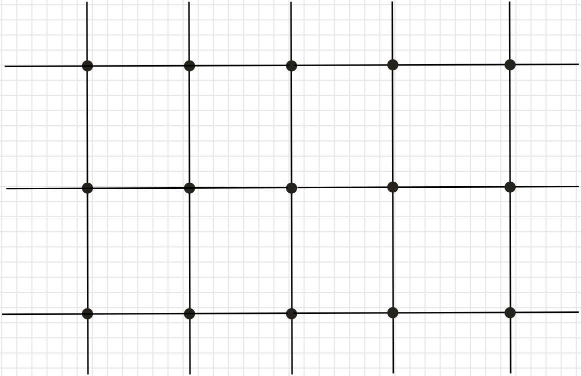

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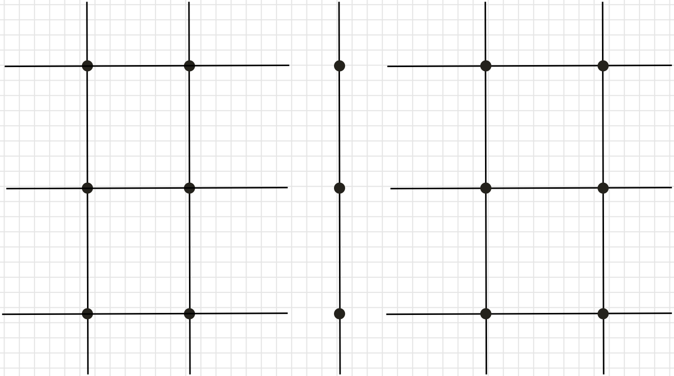

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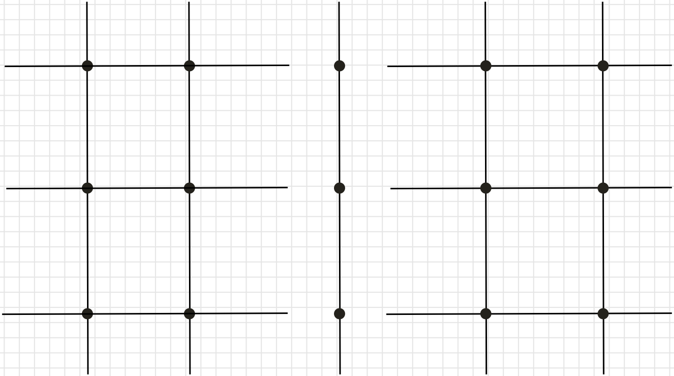
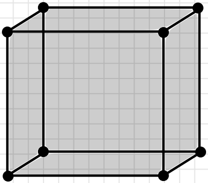


If a module has multiple parameters with maximum heights, we may need to decompose it into many pieces.

Induction on Number of Parameters

For the dimension induction, we consider a d-parameter module with height two. We want to break it into pieces with dimension less than d (and each piece we remove needs to be closed in the space containing it)



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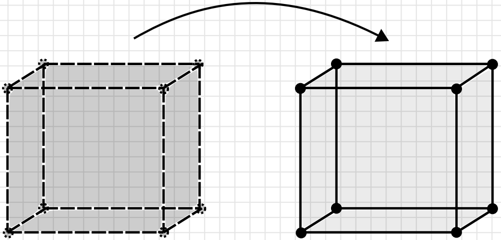

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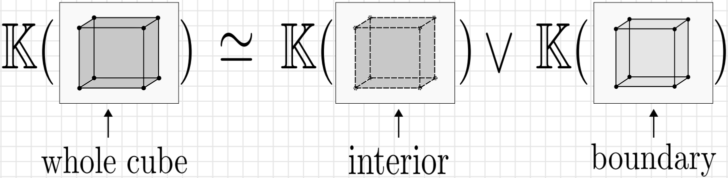

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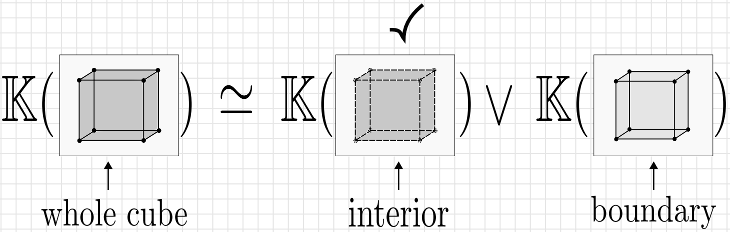

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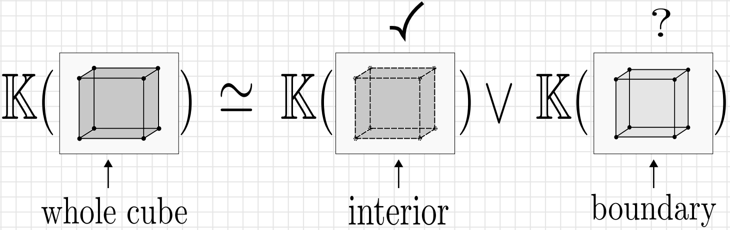

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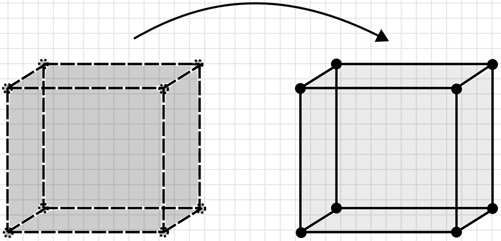
How can we break apart the boundary so that each piece looks like a parameter space with fewer than d parameters?



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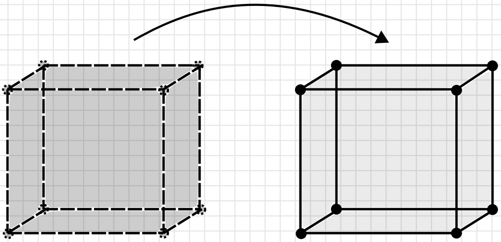

For $d = 1, 2, 3$, we can convince ourselves this is possible.
What about arbitrary finite d? Can we always remove a piece so that the remainder "falls open" into a lower-dimensional space?

Cube Unfolding

Ridge Unfoldings of Cubes (DeSplinter, Devadoss, Readyhough, 2020)

Every ridge unfolding of a finite cube will produce a net.

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- An **unfolding** of a polytope is the process of cutting along codimension-two faces so that the result can be isometrically immersed in one dimension lower.
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- Proof idea: identify a bijection between spanning trees of the dual graph of a d-cube and unfoldings, use the dual graph to show an unfolding must also be a net.

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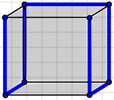
This result implies we can always remove a connected collection of codimension-two faces to leave stratified spaces with fewer than d parameters.



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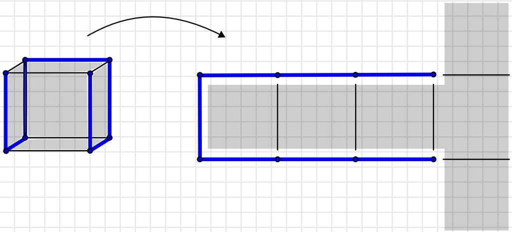
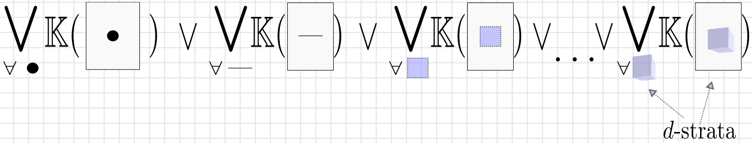


K-Theory of Multi-Parameter Persistence Modules

CLAIM:



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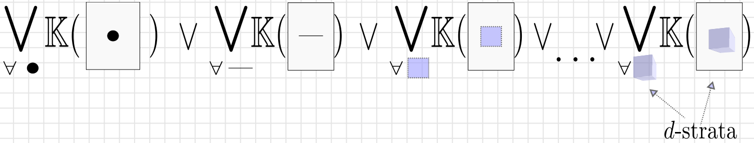

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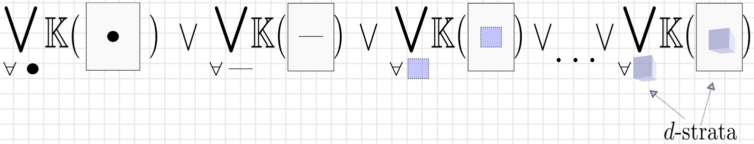

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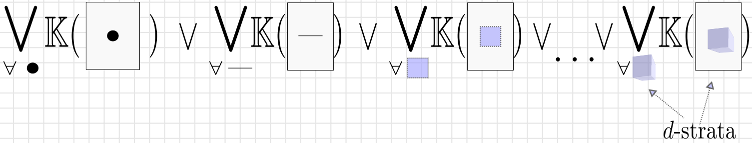

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- Remember, $\mathrm{pMod}^A(X) := \mathrm{Fun}(\mathrm{Ent}_\Delta(X), A)$.
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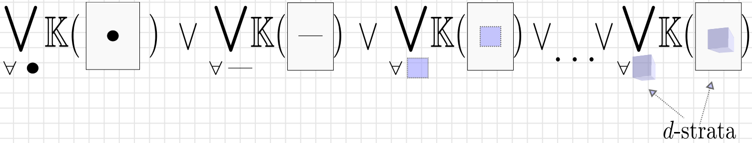

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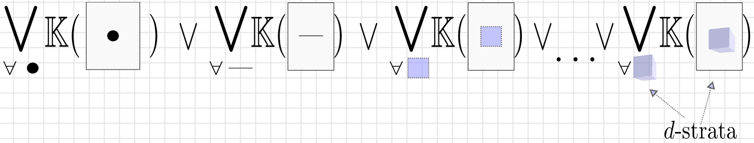

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- The K-theory depends on the target category A.

Vect-Valued Persistence Modules

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K-Theory of Multi-Parameter Zig-Zag Grid Modules (Grady, S.)

Let X be a cubical manifold embedded as a substratified space of $(\mathbb{R}^d; I)$, with finite I. There is an equivalence of spectra

$$\mathbb{K}(\mathrm{pMod}^{\mathrm{Vect}_{\mathbb{F}}}(X)) \simeq \bigvee_{x_0 \in X_0} \mathbb{K}(\mathbb{F}) \vee \bigvee_{x_1 \in X_1} \mathbb{K}(\mathbb{F}) \vee \ldots \vee \bigvee_{x_d \in X_d} \mathbb{K}(\mathbb{F})$$

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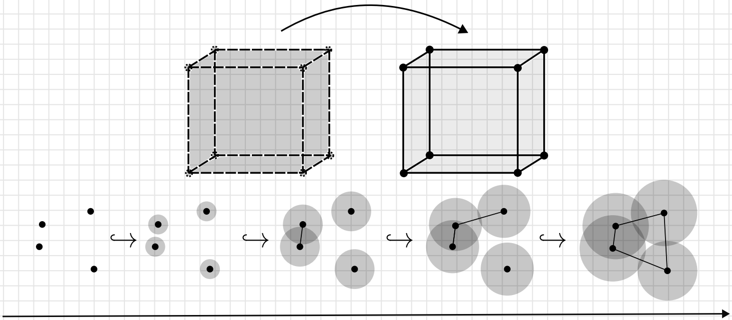
For instance, $K_0(\mathbb{F}) \cong \mathbb{Z}$, so $K_0(\mathrm{pMod}^{\mathrm{Vect}_{\mathbb{F}}}(X))$ is a direct sum of copies of $\mathbb{Z}$, one for each strata.

Next Directions

- Generalise K-theory results to persistence modules over arbitrary posets.
- Understand K_1 through natural transformations between persistence modules.

Questions?



a.k.schenfisch@tue.nl

One-parameter: https://arxiv.org/abs/2110.04591

Multi-parameter: https://arxiv.org/abs/2306.06540