Formalizing Double Categories: Univalent Double Categories

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Overview

Formalizing Double Categories

Motivation :

- Formalization valuable
- Understanding Double Categories

Challenges :

- Programming technicalities
- Univalence complicated

Why formalize categories

1. Used in mathematics:
   - Needed for geometry, algebra, topology, ...
   - Concrete example includes the Liquid Tensor Experiment.
2. Helps us understand categories:
   - In Coq UniMath we have two “notions” of categories:
     - **Categories with sets of objects:** Isomorphism invariant
     - **Univalent categories:** Equivalence invariant

Categories in Type Theory

A category can be defined in a dependent type theory as expected:

1. Type of objects $\mathcal{O} : U$
2. Type of morphisms $\mathcal{M}$ dependent on $\mathcal{O} \times \mathcal{O} \to U$
3. Identity: $\prod x : \mathcal{O}, \mathcal{M}xx$.
4. Composition: $\prod xyz : \mathcal{O}, \mathcal{M}xy \to \mathcal{M}yz \to \mathcal{M}xz$.
5. Unity: $\prod (xy : \mathcal{O})(f : x \to y), (f \circ id_x = f) \times (\mathrm{id}_y \circ f = f)$.
6. Associativity: $\prod (xyz : \mathcal{O})(f : x \to y)(g : y \to z)(h : z \to w), ((f \circ g) \circ h = f \circ (g \circ h))$.

Extensional vs. Intensional

We can construct "the universe of categories" $\mathcal{C}\mathrm{at}$ using Σ-types. What are the identities of this type?

- In Lean (an extensional TT) identities are a property and are given by strict equality.
- In Coq UniMath (an intensional TT) identities are structures. By adding key properties we get interesting universes.
  - $\mathcal{C}\mathrm{at}_{Set}$: Categories such that $\mathcal{O}$ is a set $\Rightarrow$ identities in $\mathcal{C}\mathrm{at}_{Set}$ are isomorphisms.
  - $\mathcal{C}\mathrm{at}_{Univ}$: Categories that are *univalent*[1] $\Rightarrow$ identities in $\mathcal{C}\mathrm{at}_{Univ}$ are equivalences.

Remark

We need to assume that $\mathcal{M}xy$ is a set to get the behavior we want.

[1] Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the Rezk completion. Math. Structures Comput. Sci 2015

Summary

Coq UniMath enables us to study categorical notions up to the desired notion of "sameness".

$$\mathfrak{Cat}_{(Wild)}$$

$$\mathfrak{Cat}_{Set} \Leftarrow \mathfrak{Cat}_{(Wild)} \Rightarrow \mathfrak{Cat}_{Univ}$$

Remark

"Categories that are both univalent and have type of objects sets"

||

"Univalent categories with at most one iso between two objects"

2-Categories

Let's see more examples in 2-category theory. A 2-category has:

1. Objects
2. 1-Morphisms between objects
3. Unital and associative composition of 1-morphisms
4. 2-Morphisms between 1-morphisms
5. Unital and associative composition of 2-morphisms

Classifying 2-Categories

In classical literature, 2-categories can be classified up to several layers of strictness:

(1) Isomorphism

(2) Essentially surjective & local isomorphism

(2') Equivalence of underlying category & local isomorphism of 2-morphisms

(3) Essentially surjective & local equivalence at the level of the categories formed by the 1-morphisms and 2-morphisms.

2-Categories in Coq UniMath

Following the philosophy we outlined above, what we would like are 3 characterizations of 2-categorical data with various levels of strictness, such that their identities correspond to the three classes of equivalences stated before:

1. 2-categories with a set of objects and set of 1-morphisms $2\mathfrak{C}\mathrm{at}_{set,set}$
2. 2-categories with underlying univalent 1-category $2\mathfrak{C}\mathrm{at}_{univ,set}$
3. Univalent bicategories $\mathrm{bi}\mathfrak{C}\mathrm{at}_{Univ}$.[2]

Remark

Similar to above, we want our 2-morphisms to always form sets!

[2] Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. Bicategories in univalent foundations. Math. Structures Comput. Sci. 2021

Danger! Danger!

Fact

In order to capture equivalences of 2-categories we cannot only add a univalence condition, we have to relax the whole 2-categorical structure to a bicategory (need to have non-trivial unitor and associators).

So, we get a **category theory/UniMath formalization** feedback loop:

1. Formalization can help study categorical structures up to desired equivalences.
2. This necessitates formalizing the appropriately weakened structures.
3. This needs categorical understanding of appropriate definition.

Upshot

Conclusion

Formalizing categorical structures via Coq UniMath provides a framework for studying the structure precisely up to the desired level of sameness.

This needs correctly choosing definitions that are appropriately weak!

In the Pursuit of Univalent Double Categories

The aim of our work is to develop appropriate univalence conditions for double categories! Double categories have found applications in various places

- Have been used to define limits of 2-categories[3].
- Relevant in applied category theory (Lenses, ...)[4].
- Various applications in computer science[5,6].

[3] tslil clingman and Lyne Moser. Bi-initial objects and bi-representations are not so different. Cah. Topol. Géom. Différ. Catég., 63(3):259–330, 2022

[4] Bryce Clarke. The double category of lenses. 2023.

[5] John C. Baez, Kenny Courser, and Christina Vasilakopoulou. Structured versus decorated cospans. Compositionality, 4(3):39, 2022.

[6] Pierre-Évariste Dagand and Conor McBride. A categorical treatment of ornaments. (LICS 2013)

What are double categories?

- **Idea:** A category with two types of morphisms.
- **Precise Notion:** A category object in the category of categories.
- **Explicit precise notion:**
  - Objects
  - Horizontal morphisms depending on objects
  - Vertical morphisms depending on objects
  - Squares depending on horizontal/vertical morphisms
  - Identities for objects, horizontal and vertical morphisms
  - Compositions of horizontal, vertical morphisms, squares

  - Unitality of compositions
  - Associativity of compositions

Equivalences of double categories

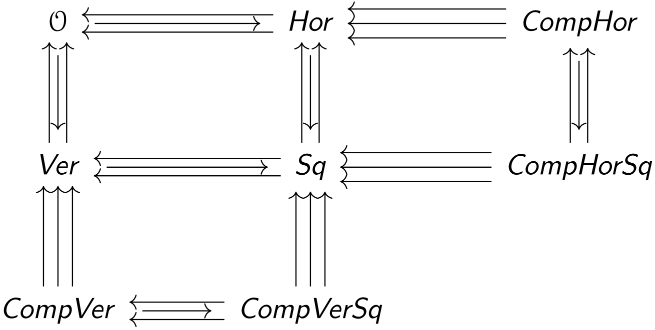
Double categories come with various notions of equivalences:

1. Isomorphisms
2. Horizontal equivalences: Equivalences on the category of objects and horizontal morphisms & the category of vertical morphisms and squares.
3. Vertical equivalences: The other way around
4. Gregarious equivalences: A double functor that is surjective on objects (up to a technical notion of equivalence), surjective on morphisms (up to isomorphism), fully faithful on squares.

Want: Notions of suitably weak double categories fitting all these types of equivalences that we can formalize!

Double categorical framework

Let's start with the basic input in type theory for double categories, which we capture via the following diagram:



We are missing unitality and associativity, and needed strictness.

Defining double categories for isomorphisms

This one is as straightforward as the ones before:

Proposition

In double categories in which:

- *The types of objects, the two morphisms and squares are sets*
- *All associativity and unitality conditions are given strictly*

the identities correspond to isomorphisms.

Defining double categories for vertical univalence

For the next one, it gets hard!

- Objects and vertical morphisms should form a univalent category.
- Horizontal morphisms and squares should be a univalent category.
- Objects and horizontal morphisms can neither be univalent nor have a set of objects... so composition has to be non-strict in general. Here we need associators and unitors.

$\Rightarrow$ We have to weaken the structure in one direction! Such definition exists under the name "pseudo-double category" by Grandis[7], Johnson–Yau[8], ...

[7] Marco Grandis. Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[8] Niles Johnson and Donald Yau. 2-dimensional categories. Oxford University Press, Oxford, 2021.

Univalent double categories for vertical equivalences.

Having all this input the claim now is that things work as one might expect that we have formalized using Coq UniMath.

Proposition (Ahrens–North–R.–van der Weide)

Identities in the universe of pseudo-double categories correspond to vertical equivalences.

Of course we have similar results if we use horizontal equivalences.

Technical sidenote

We use the method of displayed categories to actually formalize things!

- Displayed categories where introduced as a technical method to add structures, such as monoidal structures to categories.
- The approach is modular allowing for effective formalizations.
- As part of this work we generalized displayed categories to 2-sided displayed categories, which we use to formalize pseudo double categories and establish the desired univalence.

Gregarious Univalence

The next step is to develop an appropriately weak notion of double category for gregarious equivalences. Unfortunately we know very little:

1. The literature on weaker notions of double categories is very sparse. There is a notion of "double bicategory" by Verity.[9]
2. The literature on gregarious equivalences is also sparse. It was conjectured that they provide a good theory of equivalences (aka model structure), but not proven.

This points to the following conjecture we are aiming towards:

Conjecture

Identities of Verity double bicategories (or a slight variation thereof) correspond to gregarious equivalences.

[9] Dominic Verity. Enriched categories, internal categories and change of base. Repr. Theory Appl. Categ., (20):1–266, 2011.

Interlude: Double ∞-Categories

Let us make an ∞-categorical interlude, to appreciate the challenging situation.

Definition

A (non-univalent) double ∞-category is a bisimplicial space $X : \Delta^{op} \times \Delta^{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$ that satisfies injective fibrancy (technical) and Segal condition:

$$X_{kn} \xrightarrow{\simeq} X_{1n} \times_{X_{0n}} \times_{X_{0n}} X_{1n}$$

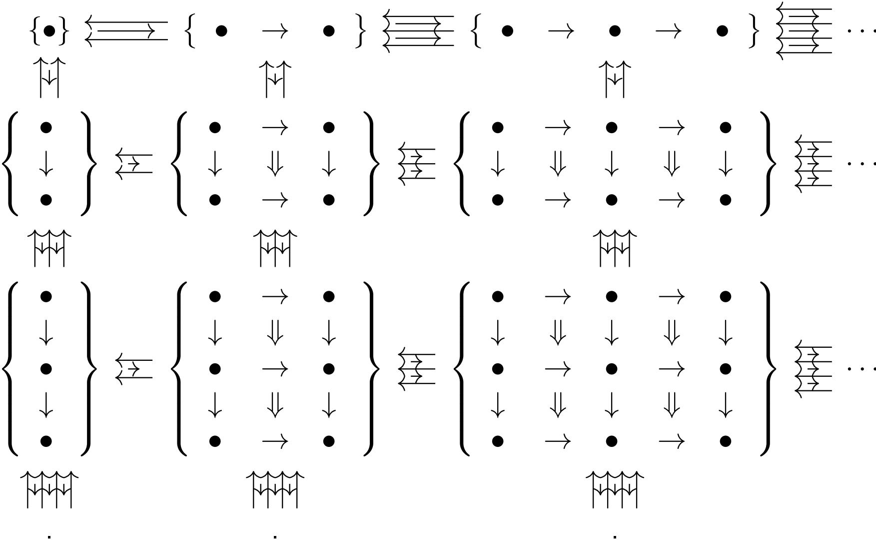
$$X_{kn} \xrightarrow{\simeq} X_{k1} \times_{X_{k0}} \times_{X_{k0}} X_{k1}$$

Question

What are univalent (complete) double ∞-categories?

The answer may surprise you! But first a diagram:

Interlude: Illustration of Double ∞-Category



Interlude: Complete Double ∞-Categories

Nobody knows for sure and there are many options around!

1. Bidirectional completeness.[10]
2. One directional completeness.[11]
3. One directional completeness + local completeness.
4. Completeness based on gregarious equivalences.[12]

Conclusion

Our project not only develops double categorical univalence, but also determines appropriate double ∞-categorical completeness!

[10] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. Adv. Math., 307:147–223, 2017

[11] Rune Haugseng. Bimodules and natural transformations for enriched ∞-categories. Homology Homotopy Appl., 18(1):71–98, 2016

[12] Alexander Campbell. Unpublished slides https://acmbl.github.io/greg_slides.pdf

Univalence Principle

The Univalence Principle[13] provides a general framework to determine correct notion of univalence in a variety of settings.

1. **Categories:** Univalence according to the principle corresponds to univalent (i.e. complete) categories, first defined by Rezk[14].

   ⇒ The structure was easy enough to determine univalence ad hoc.

2. **Double Categories:** Univalence according to the principle provides a canonical result sorting out various existing results.

   ⇒ The univalence principle is key for further progress and sorting things out among a variety of notions!

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[13] Benedikt Ahrens, Paige Randall North, Michael Shulman and Dimitris Tsementzis. The Univalence Principle, arXiv:2102.06275, to appear in Memoirs of the AMS

[14] Charles Rezk. A model for the homotopy theory of homotopy theory. Trans. Amer. Math. Soc., 353(3):973–1007, 2001.

Summary: Challenges towards gregarious Univalence

We face two major challenges:

1. On the mathematical side, the sparse literature and conflicting developments necessitates developing some of the results first.
2. On the formalization side, we cannot use 2-sided displayed categories anymore, and need more sophisticated libraries.

The End

Thank you!

- Paper: **Univalent Double Categories**, arXiv:2310.09220
  https://arxiv.org/abs/2310.09220
- Formalization:
  https://github.com/UniMath/UniMath/tree/master/UniMath/Bicategories/DoubleCategories
- Questions:
  1. Ask me in person!
  2. Email: rasekh@mpim-bonn.mpg.de