Rezk completions in formal (univalent) category theory

Yoneda structures and (2-)exactness

Kobe Wullaert

Delft University of Technology

24/11/2022

Abstract

Question

How to construct Rezk completions of categories, equipped with some (diagrammatic) structure?

Abstract

Question

How to construct Rezk completions of categories, equipped with some (diagrammatic) structure?

More precise question

How to generalize the construction(s) of the Rezk completion, internal to a (suitable) 2-category?

Abstract

Question

How to construct Rezk completions of categories, equipped with some (diagrammatic) structure?

More precise question

How to generalize the construction(s) of the Rezk completion, internal to a (suitable) 2-category?

Approach

Apply the techniques and toolkits of *formal category theory*.

Abstract

Question

How to construct Rezk completions of categories, equipped with some (diagrammatic) structure?

More precise question

How to generalize the construction(s) of the Rezk completion, internal to a (suitable) 2-category?

Approach

Apply the techniques and toolkits of *formal category theory*.

Toolkits (originate from the set-based world)

1. Yoneda structures;
2. 2-congruences and quotients.

Motivation: quote

John Gray, Adjointness for 2-Categories, 1974.

> *The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well*
>
> . . .
>
> *The basic idea is that the category of small categories, Cat, is a 2-category with properties and that one should attempt to* **identify** *those* **properties** *that enable one to* **do the "structural parts of category theory"**.

What is formal CT?

Formal category theory studies 2-categories; whose objects (resp. morphisms, and 2-cells) are *generalized* categories, functors and natural transformations, respectively.

What are *generalized* categories?

What is formal CT?

Formal category theory studies 2-categories; whose objects (resp. morphisms, and 2-cells) are *generalized* categories, functors and natural transformations, respectively.

What are *generalized* categories?

flavour: (1-)categorical structure
(ordinary) categories
enriched categories
internal categories
fibred categories
indexed categories
glued categories
monoidal categories
multicategories
0-categories (sets)
set-categories

Motivation: flavours

Formal category theory is motivated by unifying (some of) those flavours:

flavour: (1-)categorical structure	univalence	Rezk completions
(ordinary) categories	✓	✓
enriched categories	✓	✓
internal categories	-	-
fibred categories	X	X
indexed categories	?	?
glued categories	?	?
monoidal categories	✓	✓
multicategories	X	X
0-categories (sets)	:neutral-face:	:neutral-face:
set-categories	:neutral-face:	:neutral-face:

Motivation: flavours (part 2)

Category theory has different (structural) flavours:

flavour	Rezk completions
(ordinary) categories	✓
enriched categories	✓
internal categories	-
fibred categories	X
indexed categories	?
glued categories	?
monoidal categories	✓

Specialize to the presheaf (resp. quotient) construction:

Motivation: flavours (part 3)

Specialize to the (representable) presheaf construction, and (HIT-like) quotient construction:

flavour	Presheaf	Quotient
(ordinary) categories	✓	✓
enriched categories	✓	
internal categories	-	-
fibred categories	X	
indexed categories	?	
glued categories	?	
monoidal categories	✓	

What are the (structural) analogues, in the world of formal category theory?

Motivation: flavours (part 4)

What are the (structural) analogues, in the world of formal category theory?

Presheaf	flavour	Quotient
✓	(ordinary) categories	✓
✓	enriched categories	
-	internal categories	-
X	fibred categories	
?	indexed categories	
?	glued categories	
✓	monoidal categories	

Motivation: flavours (part 4)

What are the (structural) analogues, in the world of formal category theory?

Yoneda	Presheaf	flavour	Quotient	Exactness
✓	✓	(ordinary) categories	✓	✓
✓	✓	enriched categories		
-	-	internal categories	-	-
✓	?	indexed categories		✓
	?	glued categories		✓
✓	✓	monoidal categories		

Introduction

What?

A **Yoneda structure** is a *structure* on a 2-category that axiomatizes the (enriched) Yoneda embedding.

Introduction

What?

A **Yoneda structure** is a *structure* on a 2-category that axiomatizes the (enriched) Yoneda embedding.

Intuition

1. a Yoneda structure generalizes free cocompletions;
2. a 2-category with a Yoneda structure provides sufficient structure to study $\mathbf{Cat}_{\mathcal{V}}$;
3. a Yoneda structure categorifies the powerobject.

Definition

Fix a 2-category. A **Yoneda** structure consists of:

Definition

Fix a 2-category. A **Yoneda** structure consists of:

Part 1

For every object X:

1. an **object of presheaves** $\mathbb{P}(X)$;

Definition

Fix a 2-category. A **Yoneda** structure consists of:

Part 1

For every object X:

1. an **object of presheaves** $\mathbb{P}(X)$;
2. a **Yoneda morphism** $\mathcal{Y}_X : X \to \mathbb{P}(X)$;

Definition

Fix a 2-category. A **Yoneda** structure consists of:

Part 1

For every object X:

1. an **object of presheaves** $\mathbb{P}(X)$;
2. a **Yoneda morphism** $\mathcal{Y}_X : X \to \mathbb{P}(X)$;

Part 2

For every morphism $F : X \to Y$:

1. a morphism $Y(F, 1) : Y \to \mathbb{P}(X)$;
2. a 2-cell of type $\mathcal{Y}_X \Rightarrow F \cdot Y(F, 1)$,

Definition

Fix a 2-category. A **Yoneda** structure consists of:

Part 1

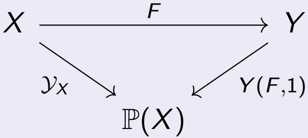
For every object X:

1. an **object of presheaves** $\mathbb{P}(X)$;
2. a **Yoneda morphism** $\mathcal{Y}_X : X \to \mathbb{P}(X)$;

Part 2

For every morphism $F : X \to Y$:

1. a morphism $Y(F,1) : Y \to \mathbb{P}(X)$;
2. a 2-cell of type $\mathcal{Y}_X \Rightarrow F \cdot Y(F,1)$,



And some (universal) properties, of course.

Universe levels

Question

How to deal with size issues?

Approaches

1. right ideal of admissible morphisms;
2. sub-2-category.

Universe levels

Question

How to deal with size issues?

Approaches

1. right ideal of admissible morphisms;
2. sub-2-category.

Advantage: admissible morphisms

Admissibility can be used to axiomatize properties (see later).

Yoneda structures: Examples and semi-representability

Examples

1. **Cat**: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$
2. **MonCat**: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_{\mathsf{Day}}$
3. $\mathbf{Cat}_{\mathcal{V}}$: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathsf{op}}, \mathcal{V}]$

Yoneda structures: Examples and semi-representability

Examples

1. **Cat**: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$
2. **MonCat**: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{Day}}$
3. $\mathbf{Cat}_{\mathcal{V}}$: $\mathbb{P}(\mathcal{C}) := [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$

Semi-representability

In the previous examples, $\mathbb{P}$ is **semi-representable**:

$$\mathbb{P}(x) := [x^{\mathrm{op}}, \Omega]$$

Open problem

Characterize (semi-)representable Yoneda structures.

Formal CT through Yoneda structures

Yoneda structures: applications

1. defines weighted (co)limits, pointwise extensions, fully faithful morphism, etc.;
2. Rezk completion is the (eso,ff)-image of the Yoneda embedding.

Formal CT through Yoneda structures

Yoneda structures: applications

1. defines weighted (co)limits, pointwise extensions, fully faithful morphism, etc.;
2. Rezk completion is the (eso,ff)-image of the Yoneda embedding.

Remark

(semi-representable) Yoneda structures provides every object with an enrichment $\rightsquigarrow$ enriched objects.

Formal CT through Yoneda structures

Yoneda structures: applications

1. defines weighted (co)limits, pointwise extensions, fully faithful morphism, etc.;
2. Rezk completion is the (eso,ff)-image of the Yoneda embedding.

Remark

(semi-representable) Yoneda structures provides every object with an enrichment $\rightsquigarrow$ enriched objects.

Remark

A Yoneda structure is not-necessarily a *universal* structure; as opposed to powerobjects.
I.e., multiple Yoneda structures.

Univalent object

Proposal

Generalize the statement: a category is univalent if and only if (small) colimits are unique, up to identity type.

Univalent object

Proposal

Generalize the statement: a category is univalent if and only if (small) colimits are unique, up to identity type.

Fix some (strict enough) 2-category, equipped with a Yoneda structure.

Definition

An object is **(Yoneda) univalent** if the type of (pointwise) left extensions into it, is a proposition.

Univalence axiom

Lemma

For every object, we have:

$$\text{representable univalence} \Rightarrow \text{(Yoneda) univalence.}$$

The converse hold if the underlying precategory is (1-)univalent.

Corollary

The sub-2-category generated by the (Yoneda) univalent objects, is univalent as a bicategory.

Question

How does univalence behave w.r.t. categories constructed via a universal property?

Rezk completions

(Universal) Definition

Let X be an (admissible) object. A **Rezk completion** of X consists of:

1. a univalent object $\mathsf{RC}(X)$;
2. an (admissible) morphism $\eta_X : X \to \mathsf{RC}(X)$;

Rezk completions

(Universal) Definition

Let X be an (admissible) object. A **Rezk completion** of X consists of:

1. a univalent object $\mathsf{RC}(X)$;
2. an (admissible) morphism $\eta_X : X \to \mathsf{RC}(X)$;

such that for every (admissible) univalent Z,

$$(\eta_X \cdot -) : \mathcal{K}(\mathsf{RC}(X), Z) \to \mathcal{K}(X, Z),$$

is an equivalence of categories.

Rezk completions

(Universal) Definition

Let X be an (admissible) object. A **Rezk completion** of X consists of:

1. a univalent object $\mathsf{RC}(X)$;
2. an (admissible) morphism $\eta_X : X \to \mathsf{RC}(X)$;

such that for every (admissible) univalent Z,

$$(\eta_X \cdot -) : \mathcal{K}(\mathsf{RC}(X), Z) \to \mathcal{K}(X, Z),$$

is an equivalence of categories.

Remark

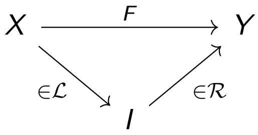
1. Rezk completions satisfy the universal property of the free univalent category;
2. we do not (yet) consider *concrete* weak equivalences.

Factorization systems

Let $\mathcal{K}$ be a 2-category. A **factorization system** consists of:

$$\mathcal{L}, \mathcal{R} : \prod_{X, Y : \mathcal{K}} \mathcal{K}(X, Y) \to \mathrm{hProp};$$

such that every morphism F is provided with a factorization



,

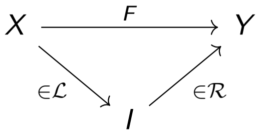
and a 2-categorical orthogonality condition.

Factorization systems

Let $\mathcal{K}$ be a 2-category. A **factorization system** consists of:

$$\mathcal{L}, \mathcal{R} : \prod_{X, Y : \mathcal{K}} \mathcal{K}(X, Y) \to \mathrm{hProp};$$

such that every morphism F is provided with a factorization



,

and a 2-categorical orthogonality condition.

Examples

Cat, **MonCat**, $\mathbf{Cat}_{\mathcal{V}}$: Eso morphisms and fully faithful morphisms

Rezk cosmos

Intuition

A **Rezk cosmos** is a 2-category, equipped with sufficient structure to mimick the presheaf construction of the Rezk completion.

Rezk cosmos

Intuition

A **Rezk cosmos** is a 2-category, equipped with sufficient structure to mimick the presheaf construction of the Rezk completion.

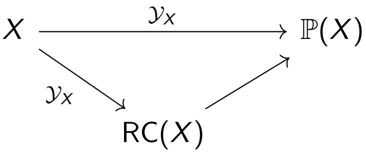
Definition

A **Rezk** cosmos consists of:

1. a 2-category $\mathcal{K}$;
2. a Yoneda structure $\mathbb{P}$; with univalent objects of presheaves;
3. a factorization system $(\mathcal{L}, \mathcal{R})$; and $\mathcal{R}$ only containing *univalence reflecting* morphisms;

Presheaf construction

1. Fix a Rezk cosmos $(\mathcal{K}, \mathbb{P}, (\mathcal{L}, \mathcal{R}))$;
2. Let X be an (admissable) morphism.
3. Denote the $(\mathcal{L}, \mathcal{R})$-factorization of $\mathcal{Y}_X$ by:



Theorem

The $\mathcal{L}$-image η_X, is a Rezk completion of X, that is: The universal property follows from the orthogonality condition.

Remarks

Remarks

1. If $\mathbb{P}$ is *good*, every (co)limit is conical.
2. We did not require cocompleteness.

Remarks

Remarks

1. If $\mathbb{P}$ is *good*, every (co)limit is conical.
2. We did not require cocompleteness.

Question

What is the relation between $(\mathcal{L}, \mathcal{R})$ and $(\mathrm{eso}, \mathrm{ff}_{\mathbb{P}})$? Does it follow from the universal property?

Examples

Motivating examples

Cat, **MonCat**, $\mathbf{Cat}_{\mathcal{V}}$

Question

How to generalize to *relative* Rezk cosmoi? In particular, pseudomonoid objects and enriched objects.

Examples

Motivating examples

Cat, **MonCat**, $\mathbf{Cat}_{\mathcal{V}}$

Question

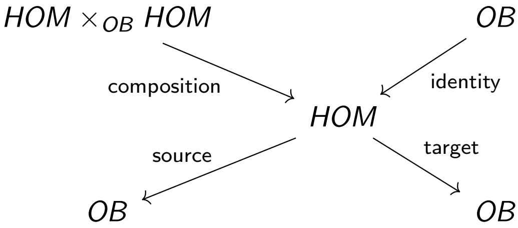
How to generalize to *relative* Rezk cosmoi? In particular, pseudomonoid objects and enriched objects.

Types of examples

1. algebras;
2. sketches;
3. (pre)sheaves;

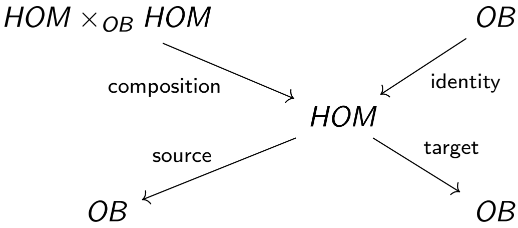
Categories as diagrams

Recall that a category is determined by the following data:



Categories as diagrams

Recall that a category is determined by the following data:



This can be defined internal to a (higher) category.

Terminology

Category objects := (suitable) internal categories.

Formal internal CT: Foundations

Set-based category theory

Category objects, internal to the 1-category of sets, form the (2-)category of set-categories: any type of objects is a set (0-type).

Formal internal CT: Foundations

Set-based category theory

Category objects, internal to the 1-category of sets, form the (2-)category of set-categories: any type of objects is a set (0-type).

Question

How to make it work in a space-based setting? More generally, what is a 2-category of internal categories?

Formal internal CT: Foundations

Set-based category theory

Category objects, internal to the 1-category of sets, form the (2-)category of set-categories: any type of objects is a set (0-type).

Question

How to make it work in a space-based setting? More generally, what is a 2-category of internal categories?

Solutions

Approximate **Cat** as:

$$\underbrace{CAT(\underbrace{\text{hGpd}}_{\text{pre}-(1,1)})}_{\textit{Weber}} \quad \underbrace{CAT(\underbrace{\mathbf{Gpd}}_{(2,1)}) \quad CAT(\underbrace{\mathbf{Cat}}_{(2,2)})}_{\textit{Street}}$$

2-congruences

Question

What is an appropriate notion of *internal categories* (or *category objects*) in a 2-category?

2-congruences

Question

What is an appropriate notion of *internal categories* (or *category objects*) in a 2-category?

Example

In **Cat**, the (bare) internal categories are double categories.

2-congruences

Question

What is an appropriate notion of *internal categories* (or *category objects*) in a 2-category?

Example

In **Cat**, the (bare) internal categories are double categories.

An internal category that *represents* an actual 1-category, is a 2-congruence:

Definition

A 2-**congruence** is an internal category if the source-target map is:

1. discrete (i.e. faithful + conservative), representably;
2. a two-sided fibration.

Quotients

A **quotient** of a 2-congruence $H \rightrightarrows O$, consists of:

1. a *quotient object* Q;
2. a *projection morphism* $q : O \to Q$;

Quotients

A **quotient** of a 2-congruence $H \rightrightarrows O$, consists of:

1. a *quotient object* Q;
2. a *projection morphism* $q : O \to Q$;

satisfying the following universal property: For any object X:

$$\mathcal{K}(Q, X) \to \mathsf{Act}(H \rightrightarrows O, X),$$

is an equivalence of categories.

Examples

Example

Let $\mathcal{C}$: **Cat** be a category.

1. every category, is an object of *objects*:

$$\mathcal{C}^{\rightarrow} \rightrightarrows \mathcal{C}$$

2. every category, is a quotient (i.e., representing object):

$$\mathrm{Core}(\mathcal{C}^{\rightarrow}) \rightrightarrows \mathrm{Core}(\mathcal{C}).$$

Examples

Example

Let $\mathcal{C}$: **Cat** be a category.

1. every category, is an object of *objects*:

$$\mathcal{C}^{\rightarrow} \rightrightarrows \mathcal{C}$$

2. every category, is a quotient (i.e., representing object):

$$\mathrm{Core}(\mathcal{C}^{\rightarrow}) \rightrightarrows \mathrm{Core}(\mathcal{C}).$$

Example

Every Street (2-)topos is exact.

Rezk completion: Quotient construction

In **Cat**, we can recover the Rezk completion as a quotient of a core groupoid:

Construction

Let $\mathcal{C}$ be a category, and $\mathrm{Core}(\mathcal{C})$ its core. The following data:

$$\mathcal{C}^{\rightarrow}|_{ob=core} \rightrightarrows \mathcal{C}$$

is part of a 2-congruence.

Rezk completion: Quotient construction

In **Cat**, we can recover the Rezk completion as a quotient of a core groupoid:

Construction

Let $\mathcal{C}$ be a category, and Core($\mathcal{C}$) its core. The following data:

$$\mathcal{C}^{\rightarrow}|_{ob=core} \rightrightarrows \mathcal{C}$$

is part of a 2-congruence.

Refer to the 2-congruence as the *core-congruence*:

Proposition

The quotient of the core-congruence, satisfies the universal property of the Rezk completion.

Observations

Recall:

Proposition

The quotient of the core-congruence, satisfies the universal property of the Rezk completion.

Observations

Recall:

Proposition

The quotient of the core-congruence, satisfies the universal property of the Rezk completion.

Proposition simplified

Any functor into a univalent category, carries a unique action on the core-congruence.

Observation

A category $\mathcal{C}$ is univalent if and only if the core-inclusion carries a unique action.

Observations

Recall:

Proposition

The quotient of the core-congruence, satisfies the universal property of the Rezk completion.

Proposition simplified

Any functor into a univalent category, carries a unique action on the core-congruence.

Observation

A category $\mathcal{C}$ is univalent if and only if the core-inclusion carries a unique action.

Remark

The Rezk completion only requires quotients of internal categories, representing certain (not-necessarily full) subcategories.

Questions

Question

How does the quotient construction, for categories, generalize to (arbitrary) 2-categories? More precisely:

1. Given a exact 2-category, with sufficient core objects. Is the core object

Given a 2-category with given core objects, satisfying some universal property.

Questions

Question

How does the quotient construction, for categories, generalize to (arbitrary) 2-categories? More precisely:

1. Given a exact 2-category, with sufficient core objects. Is the core object

Given a 2-category with given core objects, satisfying some universal property.

Question

What is the relation between:

1. the presheaf construction, via Yoneda structures;
2. the quotient construction, via exactness?

Questions

Question

How does the quotient construction, for categories, generalize to (arbitrary) 2-categories? More precisely:

1. Given a exact 2-category, with sufficient core objects. Is the core object

Given a 2-category with given core objects, satisfying some universal property.

Question

What is the relation between:

1. the presheaf construction, via Yoneda structures;
2. the quotient construction, via exactness?

In particular:

1. compare the notions of univalence;
2. is this related to representable Yoneda structures?

A conclusion

Question

What is a 2-category of internal categories, in univalent foundations?

A conclusion

Question

What is a 2-category of internal categories, in univalent foundations?

Options

1. every object is an object of *objects*, of some (canonical?)
   $\rightsquigarrow$ 2-regularity
   2-category is *embedded* into the 2-category of category objects.
2. every object is a quotient, of a (canonical?) 2-congruence

Question

Street exactness is a very strong condition. Is it possible to modify it such that:

1. exactness holds for (suitable) enriched categories?
2. compatibility between quotients and the Yoneda structure?

Question

Street exactness is a very strong condition. Is it possible to modify it such that:

1. exactness holds for (suitable) enriched categories?
2. compatibility between quotients and the Yoneda structure?

Classifying discrete opfibration

Classifying discrete opfibration, generalizes the subobject classifier:

$$\mathcal{K}(X, \Omega) \to \mathsf{DFib}(1, X).$$

Question

Street exactness is a very strong condition. Is it possible to modify it such that:

1. exactness holds for (suitable) enriched categories?
2. compatibility between quotients and the Yoneda structure?

Classifying discrete opfibration

Classifying discrete opfibration, generalizes the subobject classifier:

$$\mathcal{K}(X, \Omega) \to \mathsf{DFib}(1, X).$$

$$\begin{array}{ccc} X^{\to} & & \\ \downarrow & & \\ X \times X & \longrightarrow & \Omega \end{array}$$

The *issue* is contravariance.

Classifying category objects

A (semi-representable) Yoneda structure provides:

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \Omega$$

Question

1. Not every Yoneda structure classifies discrete opfibrations. Does it classify (contravariant) 2-congruences?
2. Can we make it work for (suitable) enriched categories?

Classifying category objects

A (semi-representable) Yoneda structure provides:

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \Omega$$

Question

1. Not every Yoneda structure classifies discrete opfibrations. Does it classify (contravariant) 2-congruences?
2. Can we make it work for (suitable) enriched categories?

Remark

Compare to 1-toposes.

Category object classifier

Proposal

Define a category object as a 2-congruence, but the source-target map is given by:

$$\sum_{(X,Y):\mathcal{C}^{\mathsf{op}}\times\mathcal{C}} \mathsf{hom}(X, Y) \rightrightarrows \mathcal{C}^{\mathsf{op}} \times \mathcal{C}.$$

Category object classifier

Proposal

Define a category object as a 2-congruence, but the source-target map is given by:

$$\sum_{(X,Y):\mathcal{C}^{\mathsf{op}}\times\mathcal{C}} \mathsf{hom}(X,Y) \rightrightarrows \mathcal{C}^{\mathsf{op}} \times \mathcal{C}.$$

We are interested in the image of a functor:

$$\mathcal{K}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \Omega) \to \mathsf{CatObj}(\mathcal{C})$$

Analogous to Weber, this induces a notion of admissible.

Category object classifier

Proposal

Define a category object as a 2-congruence, but the source-target map is given by:

$$\sum_{(X,Y):\mathcal{C}^{\mathsf{op}}\times\mathcal{C}} \mathsf{hom}(X, Y) \rightrightarrows \mathcal{C}^{\mathsf{op}} \times \mathcal{C}.$$

We are interested in the image of a functor:

$$\mathcal{K}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \Omega) \to \mathsf{CatObj}(\mathcal{C})$$

Analogous to Weber, this induces a notion of admissible.

Question

Does this characterize (semi-)representable Yoneda structures?

Conclusion

1. univalent category theory fits in the setting of formal category theory;
2. Yoneda structures and exactness are well-suited.
3. Internal categories, in univalent foundations, correspond with (2-)exactness.
4. Representability of Yoneda structure (might) correspond to exactness.