

# A 2-categorical proof of Frobenius for fibrations defined from a generic point

*DutchCATS*

*UvA, Amsterdam*

Sina Hazratpour

January 2023

This is based on a joint work with Emily Riehl at Johns Hopkins.

Preprint: **arXiv:2110.14576**

## History

In the context of models of HoTT, the Frobenius theorem says that the pushforward along fibrations preserve fibrations.

## History

In the context of models of HoTT, the Frobenius theorem says that the pushforward along fibrations preserve fibrations.

The Frobenius theorem is used for the interpretation of  $\Pi$  types.

## History

In the context of models of HoTT, the Frobenius theorem says that the pushforward along fibrations preserve fibrations.

The Frobenius theorem is used for the interpretation of  $\Pi$  types.

A non-constructive proof, using minimal fibrations, is given in the simplicial model of HoTT by Voevodsky.

## History

In the context of models of HoTT, the Frobenius theorem says that the pushforward along fibrations preserve fibrations.

The Frobenius theorem is used for the interpretation of  $\Pi$  types.

A non-constructive proof, using minimal fibrations, is given in the simplicial model of HoTT by Voevodsky.

Coquand gave a slick type theoretic proof in Cubical Type Theory.

## History

In the context of models of HoTT, the Frobenius theorem says that the pushforward along fibrations preserve fibrations.

The Frobenius theorem is used for the interpretation of  $\Pi$  types.

A non-constructive proof, using minimal fibrations, is given in the simplicial model of HoTT by Voevodsky.

Coquand gave a slick type theoretic proof in Cubical Type Theory.

Coquand's proof was analyzed using category theory by Steve Awodey and Christian Sattler.

## Our Proof: Setup

- ▶ A locally cartesian closed category  $\mathcal{E}$ . In particular, every morphism  $p: A \rightarrow X$  gives rise to an adjoint triple

$$\begin{array}{ccc} & \begin{array}{c} \xrightarrow{p_!} \\ \perp \\ \xleftarrow{p^*} \\ \perp \\ \xrightarrow{p_*} \end{array} & \\ /A & & /X \end{array}$$

- ▶ An object  $\mathbb{1}$  in  $\mathcal{E}$ .
- ▶ A class  $\text{TFib}$  of *trivial fibrations*, which
  - admit sections,
  - are stable under pushforwards (along any map),
  - are stable under retracts.



# Fibrations

We say a map  $p: A \rightarrow X$  is a **fibration** precisely when the gap map  $\delta \Rightarrow p$  is a trivial fibration.

The diagram shows a commutative square with an additional map from the top-left corner to the bottom-right corner. The nodes are arranged as follows:

- Top-left:  $A^{\square} \times \square$
- Top-right:  $A$
- Bottom-left:  $X^{\square} \times \square$
- Bottom-right:  $X$

The maps between these nodes are:

- A solid arrow from  $A^{\square} \times \square$  to  $A$  labeled  $\epsilon$ .
- A solid arrow from  $A^{\square} \times \square$  to  $X^{\square} \times \square$  labeled  $p^{\square} \times \square$ .
- A solid arrow from  $A$  to  $X$  labeled  $p$ .
- A solid arrow from  $X^{\square} \times \square$  to  $X$  labeled  $\epsilon$ .
- A dashed arrow from  $A^{\square} \times \square$  to  $A$  labeled  $\delta \Rightarrow p$ .
- A solid arrow from  $A$  to  $X^{\square} \times \square$  labeled  $\lrcorner$ .

# Frobenius Theorem

## Theorem (Coquand)

*Fibrations are closed under pushforward along other fibrations.*

---

# Frobenius Theorem

## Theorem (Coquand)

*Fibrations are closed under pushforward along other fibrations.*

In the semantics of HoTT, we interpret types by fibrations. The Frobenius theorem allows for the interpretation of  $\Pi$ -types as pushforward of fibrations along fibrations.

# Frobenius Theorem

## Theorem (Coquand)

*Fibrations are closed under pushforward along other fibrations.*

In the semantics of HoTT, we interpret types by fibrations. The Frobenius theorem allows for the interpretation of  $\Pi$ -types as pushforward of fibrations along fibrations.

$$\frac{X \vdash A \text{ Type} \quad X.A \vdash B \text{ Type}}{X \vdash \Pi_A B \text{ Type}}$$

$$\begin{array}{ccc} B & & \Pi_A B \\ q \downarrow & & \downarrow p_* q \\ A & \xrightarrow{\rho} & X \end{array}$$

## Our Proof Strategy

Our goal is to prove  $\delta \Rightarrow p_*q$  is a trivial fibration.

## Our Proof Strategy

Our goal is to prove  $\delta \Rightarrow p_*q$  is a trivial fibration.

To do this, we show  $\delta \Rightarrow p_*q$  is a retract of a pushforward of  $\delta \Rightarrow q$ , hence a trivial fibration.

## Our Proof Strategy

Our goal is to prove  $\delta \Rightarrow p_*q$  is a trivial fibration.

To do this, we show  $\delta \Rightarrow p_*q$  is a retract of a pushforward of  $\delta \Rightarrow q$ , hence a trivial fibration.

$$\begin{array}{ccccc} (\Pi_A B)^\mathbb{I} \times \mathbb{I} & \xrightarrow{\kappa} & \Pi_{A^\mathbb{I} \times \mathbb{I}} B^\mathbb{I} \times \mathbb{I} & \xrightarrow{\rho} & (\Pi_A B)^\mathbb{I} \times \mathbb{I} \\ \delta \Rightarrow p_*q \downarrow & & \downarrow (\rho^\mathbb{I} \times \mathbb{I})_*(\delta \Rightarrow q) & & \downarrow \delta \Rightarrow p_*q \\ (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\mathbb{I} \times \mathbb{I}} (B_\epsilon) & \xrightarrow{\rho_\epsilon} & (\Pi_A B)_\epsilon \end{array}$$

## Our Proof Strategy

Our goal is to prove  $\delta \Rightarrow p_*q$  is a trivial fibration.

To do this, we show  $\delta \Rightarrow p_*q$  is a retract of a pushforward of  $\delta \Rightarrow q$ , hence a trivial fibration.

$$\begin{array}{ccccc} (\Pi_A B)^\mathbb{I} \times \mathbb{I} & \xrightarrow{\kappa} & \Pi_{A^\mathbb{I} \times \mathbb{I}} B^\mathbb{I} \times \mathbb{I} & \xrightarrow{\rho} & (\Pi_A B)^\mathbb{I} \times \mathbb{I} \\ \delta \Rightarrow p_*q \downarrow & & \downarrow (p^\mathbb{I} \times \mathbb{I})_*(\delta \Rightarrow q) & & \downarrow \delta \Rightarrow p_*q \\ (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\mathbb{I} \times \mathbb{I}} (B_\epsilon) & \xrightarrow{\rho_\epsilon} & (\Pi_A B)_\epsilon \end{array}$$

To do these we use the calculus of mates from 2-category theory.



## Theorem (Kelly-Street)

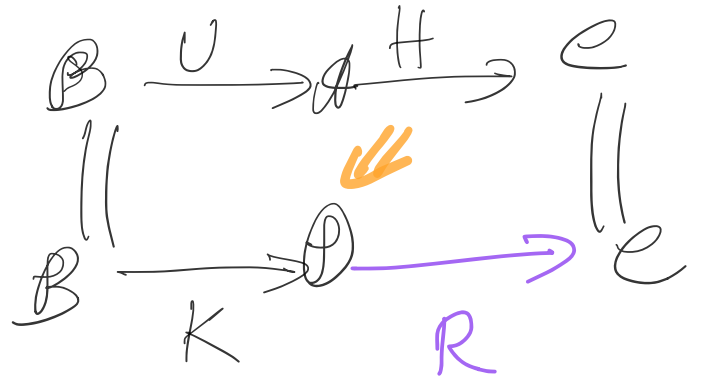
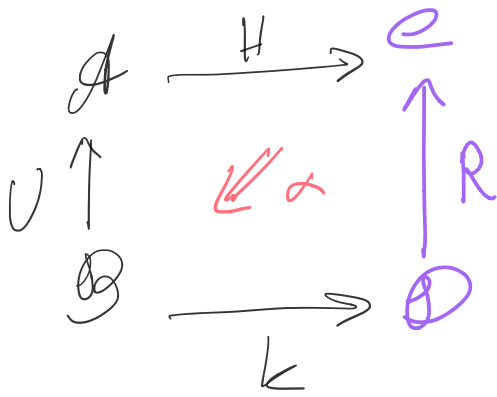
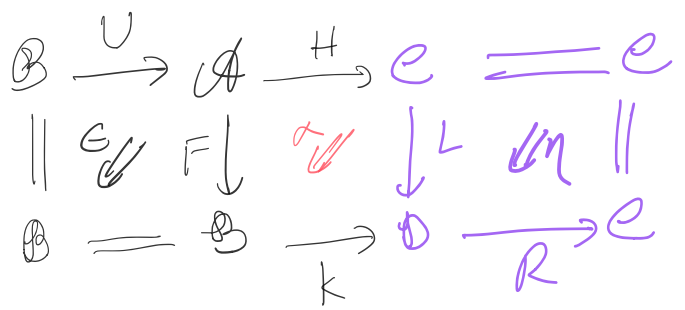
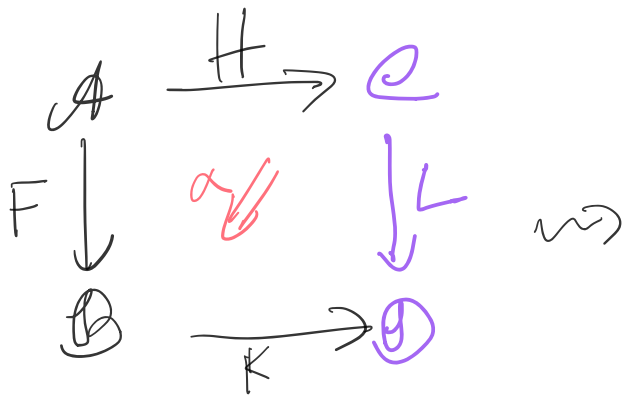
Consider the pair of double categories  $\mathbb{L}adj$  and  $\mathbb{R}adj$  whose:

- ▶ *objects are categories,*
- ▶ *horizontal arrows are functors,*
- ▶ *vertical arrows are fully-specified adjunctions pointing in the direction of the left adjoint, and*
- ▶ *squares of  $\mathbb{L}adj$  (resp.  $\mathbb{R}adj$ ) are natural transformations between the squares of functors formed by the left (resp. right) adjoints.*

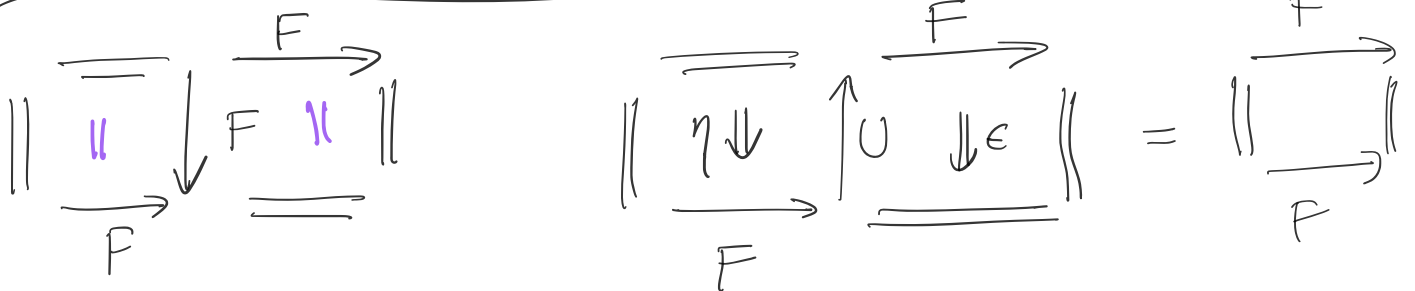
Then

$$\mathbb{L}adj \cong \mathbb{R}adj$$

*which acts on squares by taking mates.*



### Unit & Counit As Mates



# The Basic 2-cells

From the counit 2-cells

$$\begin{array}{ccc} /0 & \xrightarrow{\eta_!} & /1 \\ \eta^* \uparrow & \Downarrow \pi & \parallel \\ /1 & \xlongequal{\quad} & /1 \end{array} \qquad \begin{array}{ccc} /1 & \xrightarrow{\eta^*} & /0 \\ \eta_* \uparrow & \Downarrow \nu & \parallel \\ /0 & \xlongequal{\quad} & /0 \end{array}$$

# The Basic 2-cells

From the counit 2-cells

$$\begin{array}{ccc}
 \begin{array}{c} \lrcorner \\ \downarrow \\ \lrcorner \end{array} & \xrightarrow{\mathbb{I}_X^*} & \begin{array}{c} \lrcorner \\ \downarrow \\ \lrcorner \end{array} \\
 & & \xrightarrow{\mathbb{I}_X} \\
 & & /1 \\
 & & \xrightarrow{\mathbb{I}_!} \\
 & & /1
 \end{array}
 \quad
 \begin{array}{ccc}
 /0 & \xrightarrow{\mathbb{I}_!} & /1 \\
 \mathbb{I}_* \uparrow & \downarrow \pi & \parallel \\
 /1 & \xlongequal{\quad} & /1
 \end{array}
 \quad
 \begin{array}{ccc}
 /1 & \xrightarrow{\mathbb{I}^*} & /0 \\
 \mathbb{I}_* \uparrow & \downarrow \nu & \parallel \\
 /0 & \xlongequal{\quad} & /0
 \end{array}$$

we obtain the span

$$X^0 \xleftarrow{\pi} X^0 \times 0 \xrightarrow{\epsilon} X$$

natural in  $X$ .

## The Basic 2-cells

From the counit 2-cells

$$\begin{array}{ccc}
 / \mathbb{1} & \xrightarrow{\mathbb{1}_!} & / \mathbf{1} \\
 \mathbb{1}^* \uparrow & \Downarrow \pi & \parallel \\
 / \mathbf{1} & \xlongequal{\quad} & / \mathbf{1}
 \end{array}
 \qquad
 \begin{array}{ccc}
 / \mathbf{1} & \xrightarrow{\mathbb{1}^*} & / \mathbb{1} \\
 \mathbb{1}^* \uparrow & \Downarrow \nu & \parallel \\
 / \mathbb{1} & \xlongequal{\quad} & / \mathbb{1}
 \end{array}$$

we obtain the span

$$\mathbf{X}^{\mathbb{1}} \xleftarrow{\pi} \mathbf{X}^{\mathbb{1}} \times \mathbb{1} \xrightarrow{\epsilon} \mathbf{X}$$

natural in  $\mathbf{X}$ .

Moreover,  $\pi$  is cartesian:

$$\begin{array}{ccc}
 \mathbf{A}^{\mathbb{1}} \times \mathbb{1} & \xrightarrow{\pi} & \mathbf{A}^{\mathbb{1}} \\
 \rho^{\mathbb{1}} \times \mathbb{1} \downarrow & \lrcorner & \downarrow \rho^{\mathbb{1}} \\
 \mathbf{X}^{\mathbb{1}} \times \mathbb{1} & \xrightarrow{\pi} & \mathbf{X}^{\mathbb{1}}
 \end{array}$$

## Leibniz Exponential from the Basic 2-cells

The component of the whiskered counit

$$\begin{array}{ccc}
 \overset{A^{\mathbb{I}}}{/X^{\mathbb{I}}} & \xrightarrow{\pi^*} & /X^{\mathbb{I}} \times \mathbb{I} \\
 \pi_* \uparrow & & \downarrow \nu \quad \parallel \\
 \overset{A}{/X} & \xrightarrow{\epsilon^*} & /X^{\mathbb{I}} \times \mathbb{I} = /X^{\mathbb{I}} \times \mathbb{I}
 \end{array}$$

$A^{\mathbb{I}}_{\times \mathbb{I}}$

at  $p: A \rightarrow X$  is the Leibniz exponential  $\delta \Rightarrow p: A^{\mathbb{I}} \times \mathbb{I} \rightarrow A_{\epsilon}$ .

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc} A^\perp \times \mathbb{1} & \xrightarrow{\epsilon} & A \\ p^\perp \times \mathbb{1} \downarrow & & \downarrow p \\ X^\perp \times \mathbb{1} & \xrightarrow{\epsilon} & X \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc} /A^\mathbb{0} \times \mathbb{0} & \xrightarrow{\epsilon_!} & /A \\ (p^\mathbb{0} \times \mathbb{0})_! \downarrow & & \downarrow p_! \\ /X^\mathbb{0} \times \mathbb{0} & \xrightarrow{\epsilon_!} & /X \end{array}$$



## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc} /A^\mathbb{I} \times \mathbb{I} & \xrightarrow{\epsilon!} & /A \\ (p^\mathbb{I} \times \mathbb{I})^* \uparrow & \Downarrow & \uparrow p^* \\ /X^\mathbb{I} \times \mathbb{I} & \xrightarrow{\epsilon!} & /X \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc} /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\ (p^\square \times \square)^* \uparrow & \cong & \uparrow p^* \\ /X^\square \times \square & \xleftarrow{\epsilon^*} & /X \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc} /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\ (p^\square \times \square)_* \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\ /X^\square \times \square & \xleftarrow{\epsilon^*} & /X \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc}
 \mathbf{A}^\square \times \square & \xrightarrow{\epsilon} & \mathbf{A} \\
 p^\square \times \square \downarrow & & \downarrow p \\
 \mathbf{X}^\square \times \square & \xrightarrow{\epsilon} & \mathbf{X}
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 / \mathbf{A}^\square \times \square & \xrightarrow{\epsilon!} & / \mathbf{A} \\
 (p^\square \times \square)! \downarrow & & \downarrow p! \\
 / \mathbf{X}^\square \times \square & \xrightarrow{\epsilon!} & / \mathbf{X}
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 / \mathbf{A}^\square \times \square & \xleftarrow{\epsilon^*} & / \mathbf{A} \\
 (p^\square \times \square)_* \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\
 / \mathbf{X}^\square \times \square & \xleftarrow{\epsilon^*} & / \mathbf{X}
 \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A^\square \times \mathbb{1} & \xrightarrow{\epsilon} & A \\
 p^\square \times \mathbb{1} \downarrow & & \downarrow p \\
 X^\square \times \mathbb{1} & \xrightarrow{\epsilon} & X
 \end{array} & \rightsquigarrow & \begin{array}{ccc}
 /A^\square \times \mathbb{1} & \xrightarrow{\epsilon!} & /A \\
 (p^\square \times \mathbb{1})! \downarrow & & \downarrow p! \\
 /X^\square \times \mathbb{1} & \xrightarrow{\epsilon!} & /X
 \end{array} \\
 & & \rightsquigarrow & \begin{array}{ccc}
 /A^\square \times \mathbb{1} & \xleftarrow{\epsilon^*} & /A \\
 (p^\square \times \mathbb{1})_* \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\
 /X^\square \times \mathbb{1} & \xleftarrow{\epsilon^*} & /X
 \end{array}
 \end{array}$$

The component of  $\kappa_\epsilon$  at  $q: B \rightarrow A$  defines a map  $\kappa_\epsilon: (\Pi_A B)_\epsilon \rightarrow \Pi_{A^\square \times \mathbb{1}} B_\epsilon$  over  $X^\square \times \mathbb{1}$ .

## Constructing $\kappa_\epsilon$ via Mates

So far,

$$\begin{array}{ccc} (\Pi_A B)^\perp \times \mathbb{I} & & \Pi_{A^\perp \times \mathbb{I}} B^\perp \times \mathbb{I} \\ \delta \Rightarrow p_* q \downarrow & & \downarrow (p^\perp \times \mathbb{I})_* (\delta \Rightarrow q) \\ (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\perp \times \mathbb{I}} (B_\epsilon) \end{array}$$

## Constructing $\kappa_\epsilon$ via Mates

So far,

$$\begin{array}{ccc} (\Pi_A B)^\emptyset \times \mathbb{I} & \overset{\kappa}{\dashrightarrow} & \Pi_{A^\emptyset \times \mathbb{I}} B^\emptyset \times \mathbb{I} \\ \delta \Rightarrow p_* q \downarrow & & \downarrow (p^\emptyset \times \mathbb{I})_*(\delta \Rightarrow q) \\ (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\emptyset \times \mathbb{I}} (B_\epsilon) \end{array}$$

Next, we find the top arrow.

# Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccccccc}
 /A^\square \times \square & \xleftarrow{\pi^*} & /A^\square & \xleftarrow{\pi_*} & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 \downarrow (p^\square \times \square)_* & \cong & \downarrow p_*^\square & \cong & \downarrow (p^\square \times \square)_* & \uparrow \kappa_\epsilon & \downarrow p_* \\
 /X^\square \times \square & \xleftarrow{\pi^*} & /X^\square & \xleftarrow{\pi_*} & /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}$$



## Constructing $\kappa$ from $\kappa_\epsilon$

The component of this composite 2-cell at  $q: B \rightarrow A$  defines a map

$$\kappa: (\Pi_A B)^\natural \times \mathbb{1} \rightarrow \Pi_{A^\natural \times \mathbb{1}}(B^\natural \times \mathbb{1})$$

over  $X^\natural \times \mathbb{1}$ .

## Constructing $\kappa$ from $\kappa_\epsilon$

The component of this composite 2-cell at  $q: B \rightarrow A$  defines a map

$$\kappa: (\Pi_A B)^\square \times \square \rightarrow \Pi_{A^\square \times \square}(B^\square \times \square)$$

over  $X^\square \times \square$ .

$$\begin{array}{ccccccc}
 B^\square \times \square / A^\square \times \square & \xleftarrow{\pi^*} & B^\square / A^\square & \xleftarrow{\pi_*} & B_\epsilon / A^\square \times \square & \xleftarrow{\epsilon^*} & B / A \\
 \downarrow (p^\square \times \square)_* & & & & & & \downarrow p_* \\
 (\Pi_A B)^\square \times \square / X^\square \times \square & \xleftarrow{\pi^*} & (\Pi_A B)^\square / X^\square & \xleftarrow{\pi_*} & (\Pi_A B)_\epsilon / X^\square \times \square & \xleftarrow{\epsilon^*} & \Pi_A B / X
 \end{array}$$

## Constructing $\kappa$ from $\kappa_\epsilon$

We now have to verify that the square below commutes.

$$\begin{array}{ccc} (\Pi_A B)^\mathbb{I} \times \mathbb{I} & \xrightarrow{\kappa} & \Pi_{A^\mathbb{I} \times \mathbb{I}} B^\mathbb{I} \times \mathbb{I} \\ \delta \Rightarrow p_* q \downarrow & & \downarrow (p^\mathbb{I} \times \mathbb{I})_* (\delta \Rightarrow q) \\ (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\mathbb{I} \times \mathbb{I}} (B_\epsilon) \end{array}$$

# Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccccc}
 /A^\square \times \square & \xlongequal{\quad} & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 (p^\square \times \square)_* \downarrow & & (p^\square \times \square)_* \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\
 /X^\square \times \square & \xlongequal{\quad} & /X^\square \times \square & \xleftarrow{\epsilon^*} & /X \\
 \parallel & \uparrow \nu & \parallel & & \parallel \\
 /X^\square \times \square & \xleftarrow{\pi^*} & /X^\square & \xleftarrow{\pi_*} & /X^\square \times \square \xleftarrow{\epsilon^*} /X
 \end{array}$$

$$\begin{array}{ccccccc}
 /A^\square \times \square & \xlongequal{\quad} & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 \parallel & & \uparrow \nu & & \parallel & & \parallel \\
 /A^\square \times \square & \xleftarrow{\pi^*} & /A^\square & \xleftarrow{\pi_*} & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 (p^\square \times \square)_* \downarrow & \cong & \downarrow & \cong & \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\
 /X^\square \times \square & \xleftarrow{\pi^*} & /X^\square & \xleftarrow{\pi_*} & /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}$$

# Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccc}
 /A^\emptyset \times \emptyset & \xlongequal{\quad} & /A^\emptyset \times \emptyset \\
 \downarrow (p^\emptyset \times \emptyset)_* & & \downarrow (p^\emptyset \times \emptyset)_* \\
 /X^\emptyset \times \emptyset & \xlongequal{\quad} & /X^\emptyset \times \emptyset \\
 \parallel & \uparrow \nu & \parallel \\
 /X^\emptyset \times \emptyset & \xleftarrow{\pi^*} /X^\emptyset & \xleftarrow{\pi_*} /X^\emptyset \times \emptyset
 \end{array}
 \quad \stackrel{?}{=} \quad
 \begin{array}{ccc}
 /A^\emptyset \times \emptyset & \xlongequal{\quad} & /A^\emptyset \times \emptyset \\
 \parallel & & \parallel \\
 /A^\emptyset \times \emptyset & \xleftarrow{\pi^*} /A^\emptyset & \xleftarrow{\pi_*} /A^\emptyset \times \emptyset \\
 \downarrow (p^\emptyset \times \emptyset)_* & \cong & \downarrow (p^\emptyset \times \emptyset)_* \\
 /X^\emptyset \times \emptyset & \xleftarrow{\pi^*} /X^\emptyset & \xleftarrow{\pi_*} /X^\emptyset \times \emptyset
 \end{array}$$

# Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccccc}
 /A^\square \times \square & \xrightarrow{\pi_!} & /A^\square & \xrightarrow{\pi^*} & /A^\square \times \square \\
 \parallel & & \uparrow \iota & & \parallel \\
 /A^\square \times \square & \xlongequal{\quad} & & & /A^\square \times \square \\
 (p^\square \times \square)_! \downarrow & & & & \downarrow (p^\square \times \square)_! \\
 /X^\square \times \square & \xlongequal{\quad} & & & /X^\square \times \square
 \end{array}$$

$$\begin{array}{ccccc}
 /A^\square \times \square & \xrightarrow{\pi_!} & /A^\square & \xrightarrow{\pi^*} & /A^\square \times \square \\
 (p^\square \times \square)_! \downarrow & & \downarrow p^\square & \cong & \downarrow (p^\square \times \square)_! \\
 ? \quad /X^\square \times \square & \xrightarrow{\pi_!} & /X^\square & \xrightarrow{\pi^*} & /X^\square \times \square \\
 \parallel & & \uparrow \iota & & \parallel \\
 /X^\square \times \square & \xlongequal{\quad} & & & /X^\square \times \square
 \end{array}$$

# Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccccc}
 /A^\square \times \square & \xrightarrow{\pi_!} & /A^\square & \xrightarrow{p_!^\square} & /X^\square \\
 \parallel & \uparrow \iota & \downarrow \pi^* & \cong & \downarrow \pi^* \\
 /A^\square \times \square & \xlongequal{\quad} & /A^\square \times \square & \xrightarrow{(p^\square \times \square)_!} & /X^\square \times \square
 \end{array}$$

$$\begin{array}{ccccc}
 ? \\
 /A^\square \times \square & \xrightarrow{(p^\square \times \square)_!} & /X^\square \times \square & \xrightarrow{\pi_!} & /X^\square \\
 \parallel & & \parallel & \uparrow \iota & \downarrow \pi^* \\
 /A^\square \times \square & \xrightarrow{(p^\square \times \square)_!} & /X^\square \times \square & \xlongequal{\quad} & /X^\square \times \square
 \end{array}$$

## Constructing $\kappa$ from $\kappa_\epsilon$

$$\begin{array}{ccccc}
 /A^\square \times \square & \xrightarrow{\pi_\square} & /A^\square & \xrightarrow{p_\square^\square} & /X^\square \\
 \parallel & & \pi_\square \uparrow & & \pi_\square \uparrow \\
 /A^\square \times \square & \xlongequal{\quad} & /A^\square \times \square & \xrightarrow{(p^\square \times \square)_\square} & /X^\square \times \square
 \end{array}
 =
 \begin{array}{ccccc}
 /A^\square \times \square & \xrightarrow{(p^\square \times \square)_\square} & /X^\square \times \square & \xrightarrow{\pi_\square} & /X^\square \\
 \parallel & & \parallel & & \pi_\square \uparrow \\
 /A^\square \times \square & \xrightarrow{(p^\square \times \square)_\square} & /X^\square \times \square & \xlongequal{\quad} & /X^\square \times \square
 \end{array}$$



# Constructing the Retract $\rho_\epsilon$

Since  $p$  is a fibration,  $\delta \Rightarrow p$  is a trivial fibration and therefore it has a section:

$$\begin{array}{ccc}
 A^\square \times \square & \xrightarrow{\epsilon} & A \\
 \delta \Rightarrow p \swarrow & & \downarrow p \\
 A_\epsilon & \xrightarrow{p^* \epsilon} & A \\
 \epsilon^* p \downarrow \lrcorner & & \\
 X^\square \times \square & \xrightarrow{\epsilon} & X \\
 p^\square \times \square \swarrow & & \\
 & & 
 \end{array}$$
  

$$\begin{array}{ccc}
 /A_\epsilon & \xleftarrow{(p^* \epsilon)^*} & /A \\
 \sigma^* \uparrow & \cong & \parallel \\
 /A^\square \times \square & \xleftarrow{\epsilon^*} & /A
 \end{array}$$

## Constructing the Retract $\rho_\epsilon$

$$\begin{array}{ccc}
 /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 (\epsilon^*p)^* \uparrow & \cong & \uparrow p^* \\
 /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}
 =
 \begin{array}{ccccc}
 /A_\epsilon & \xlongequal{\quad} & /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 \uparrow & & \sigma^* \uparrow & \cong & \parallel \\
 (\epsilon^*p)^* \uparrow & \cong & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 & & (p^\square \times \square)^* \uparrow & \cong & \uparrow p^* \\
 /X^\square \times \square & \xlongequal{\quad} & /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}$$

# Constructing the Retract $\rho_\epsilon$

$$\begin{array}{ccc}
 /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 (\epsilon^*p)_* \downarrow & \cong & \downarrow p_* \\
 /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}
 =
 \begin{array}{ccccc}
 /A_\epsilon & \xlongequal{\quad} & /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 \downarrow (\epsilon^*p)_* & & \sigma_* \downarrow & \uparrow \tau & \parallel \\
 & \cong & /A^\square \times \square & \xleftarrow{\epsilon^*} & /A \\
 & & (p^\square \times \square)_* \downarrow & \uparrow \kappa_\epsilon & \downarrow p_* \\
 /X^\square \times \square & \xlongequal{\quad} & /X^\square \times \square & \xleftarrow{\epsilon^*} & /X
 \end{array}$$

# Constructing the Retract $\rho_\epsilon$

$$\begin{array}{c}
 /X \xleftarrow{p_*} /A \\
 \epsilon^* \downarrow \quad = \quad \downarrow p_* \\
 /X^\square \times \square \xleftarrow{\epsilon^*} /X
 \end{array}
 =
 \begin{array}{c}
 /X \xleftarrow{p_*} /A \\
 \parallel \\
 /A_\epsilon \xleftarrow{(p^*\epsilon)^*} /A \\
 \downarrow \sigma_* \quad \uparrow \tau_\epsilon \\
 /X^\square \times \square \xleftarrow{\epsilon^*} /A \\
 \downarrow (p^\square \times \square)_* \quad \uparrow \kappa_\epsilon \\
 /X^\square \times \square \xleftarrow{\epsilon^*} /X
 \end{array}$$

The diagram illustrates the construction of the retract  $\rho_\epsilon$  through a series of commutative diagrams and isomorphisms. The leftmost diagram shows a square with  $/X \xleftarrow{p_*} /A$  at the top,  $/X^\square \times \square \xleftarrow{\epsilon^*} /X$  at the bottom, and vertical arrows  $\epsilon^*$  and  $p_*$ . This is equated to a larger diagram where the top row is  $/X \xleftarrow{p_*} /A$  and the bottom row is  $/X^\square \times \square \xleftarrow{\epsilon^*} /X$ . The left vertical arrow is  $\epsilon^*$ . The right vertical arrow is  $p_*$ . The middle part of the diagram consists of several rows and columns of objects and maps:

- Top row:  $/X \xleftarrow{p_*} /A$
- Second row:  $/A_\epsilon \xleftarrow{(p^*\epsilon)^*} /A$
- Third row:  $/A^\square \times \square \xleftarrow{\epsilon^*} /A$
- Bottom row:  $/X^\square \times \square \xleftarrow{\epsilon^*} /X$

The maps between these rows and columns are:

- Horizontal maps:  $/X \xleftarrow{p_*} /A$ ,  $/A_\epsilon \xleftarrow{(p^*\epsilon)^*} /A$ ,  $/A^\square \times \square \xleftarrow{\epsilon^*} /A$ ,  $/X^\square \times \square \xleftarrow{\epsilon^*} /X$ .
- Vertical maps:  $/X \xrightarrow{\epsilon^*} /X^\square \times \square$ ,  $/A \xrightarrow{p_*} /X$ ,  $/A_\epsilon \xrightarrow{\sigma_*} /A^\square \times \square$ ,  $/A^\square \times \square \xrightarrow{(p^\square \times \square)_*} /X^\square \times \square$ ,  $/A \xrightarrow{\tau_\epsilon} /A_\epsilon$ ,  $/X \xrightarrow{\kappa_\epsilon} /X^\square \times \square$ .
- Isomorphisms:  $/X \cong /A_\epsilon$ ,  $/A_\epsilon \cong /A^\square \times \square$ ,  $/A^\square \times \square \cong /X^\square \times \square$ .

## Constructing the Retract $\rho_\epsilon$

$$\begin{array}{ccccc}
 (\Pi_A B)^\mathbb{I} \times \mathbb{I} & \xrightarrow{\kappa} & \Pi_{A^\mathbb{I} \times \mathbb{I}} B^\mathbb{I} \times \mathbb{I} & & (\Pi_A B)^\mathbb{I} \times \mathbb{I} \\
 \delta \Rightarrow p_* q \downarrow & & \downarrow (p^\mathbb{I} \times \mathbb{I})_*(\delta \Rightarrow q) & & \downarrow \delta \Rightarrow p_* q \\
 (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\mathbb{I} \times \mathbb{I}} (B_\epsilon) & \xrightarrow{\rho_\epsilon} & (\Pi_A B)_\epsilon
 \end{array}$$

## Constructing $\rho$ from $\rho_\epsilon$

$$\begin{array}{ccccccc}
 /A^\square \times \mathbb{1} & \xleftarrow{\pi^*} & /A^\square & \xleftarrow{\pi_*} & /A^\square \times \mathbb{1} & \xleftarrow{\epsilon^*} & /A \\
 (p^\square \times \mathbb{1})_* \downarrow & \cong & p_*^\square & \cong & (p^\square \times \mathbb{1})_* & \Downarrow \rho_\epsilon & \downarrow p_* \\
 /X^\square \times \mathbb{1} & \xleftarrow{\pi^*} & /X^\square & \xleftarrow{\pi_*} & /X^\square \times \mathbb{1} & \xleftarrow{\epsilon^*} & /X
 \end{array}$$

## Completing the Proof

Similar to the commutativity of the square involving  $\kappa_\epsilon$  and  $\kappa$  we show that the following square commutes:

$$\begin{array}{ccc}
 \Pi_{A^\flat \times \mathbb{1}} B^\flat \times \mathbb{1} & \xrightarrow{\rho} & (\Pi_A B)^\flat \times \mathbb{1} \\
 (\rho^\flat \times \mathbb{1})_*(\delta \Rightarrow q) \downarrow & & \downarrow \delta \Rightarrow \rho_* q \\
 \Pi_{A^\flat \times \mathbb{1}}(B_\epsilon) & \xrightarrow{\rho_\epsilon} & (\Pi_A B)_\epsilon
 \end{array}$$

That  $\rho$  is a retract of  $\kappa$  follows from the fact that  $\rho_\epsilon$  is a retract of  $\kappa_\epsilon$  and the iso 2-cells pasted to the left of  $\kappa_\epsilon$  and  $\rho_\epsilon$ , respectively, are pairwise inverses.

$$\begin{array}{ccccc}
 (\Pi_A B)^\flat \times \mathbb{1} & \xrightarrow{\kappa} & \Pi_{A^\flat \times \mathbb{1}} B^\flat \times \mathbb{1} & \xrightarrow{\rho} & (\Pi_A B)^\flat \times \mathbb{1} \\
 \delta \Rightarrow \rho_* q \downarrow & & \downarrow (\rho^\flat \times \mathbb{1})_*(\delta \Rightarrow q) & & \downarrow \delta \Rightarrow \rho_* q \\
 (\Pi_A B)_\epsilon & \xrightarrow{\kappa_\epsilon} & \Pi_{A^\flat \times \mathbb{1}}(B_\epsilon) & \xrightarrow{\rho_\epsilon} & (\Pi_A B)_\epsilon
 \end{array}$$