A 2-categorical proof of Frobenius for fibrations defined from a generic point

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January 2023

This is based on a joint work with Emily Riehl at Johns Hopkins.

Preprint: **arXiv:2110.14576**

History

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Coquand's proof was analyzed using category theory by Steve Awodey and Christian Sattler.

Our Proof: Setup

- A locally cartesian closed category $\mathscr{E}$. In particular, every morphism $p\colon A \to X$ gives rise to an adjoint triple

$$/A \;\; \overset{p_!}{\underset{p_*}{\underset{\perp}{\overset{\perp}{\leftarrow p^* -}}}} \;\; /X$$

- An object ▯ in $\mathscr{E}$.
- A class TFib of *trivial fibrations*, which

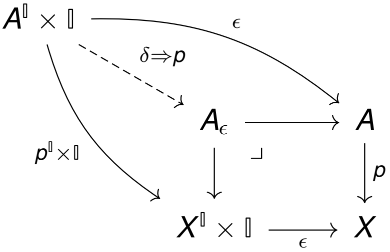
  admit sections,

  are stable under pushforwards (along any map),

  are stable under retracts.

Fibrations

We say a map $p\colon A \to X$ is a **fibration** precisely when the gap map $\delta \Rightarrow p$ is a trivial fibration.



Frobenius Theorem

Theorem (Coquand)

Fibrations are closed under pushforward along other fibrations.

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Fibrations are closed under pushforward along other fibrations.

In the semantics of HoTT, we interpret types by fibrations. The Frobenius theorem allows for the interpretation of Π-types as pushforward of fibrations along fibrations.

$$\frac{X \vdash A \ \mathrm{Type} \qquad X.A \vdash B \ \mathrm{Type}}{X \vdash \Pi_A B \ \mathrm{Type}}$$

$$\begin{array}{ccc} B & & \Pi_A B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p_* q} \\ A & \xrightarrow[p]{} & X \end{array}$$

Our Proof Strategy

Our goal is to prove $\delta \Rightarrow p_* q$ is a trivial fibration.

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$$
\begin{array}{ccccc}
(\Pi_A B)^{\square} \times \square & \xrightarrow{\kappa} & \Pi_{A^{\square} \times \square} B^{\square} \times \square & \xrightarrow{\rho} & (\Pi_A B)^{\square} \times \square \\
{\scriptstyle \delta \Rightarrow p_* q} \downarrow & & \downarrow {\scriptstyle (p^{\square} \times \square)_* (\delta \Rightarrow q)} & & \downarrow {\scriptstyle \delta \Rightarrow p_* q} \\
(\Pi_A B)_\epsilon & \xrightarrow[\kappa_\epsilon]{} & \Pi_{A^{\square} \times \square} (B_\epsilon) & \xrightarrow[\rho_\epsilon]{} & (\Pi_A B)_\epsilon
\end{array}
$$

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{\scriptstyle \delta \Rightarrow p_* q} \big\downarrow & & \big\downarrow {\scriptstyle (p^{\square} \times \square)_* (\delta \Rightarrow q)} & & \big\downarrow {\scriptstyle \delta \Rightarrow p_* q} \\
(\Pi_A B)_\epsilon & \xrightarrow[\kappa_\epsilon]{} & \Pi_{A^{\square} \times \square} (B_\epsilon) & \xrightarrow[\rho_\epsilon]{} & (\Pi_A B)_\epsilon
\end{array}
$$

To do these we use the calculus of mates from 2-category theory.

Theorem (Kelly-Street)

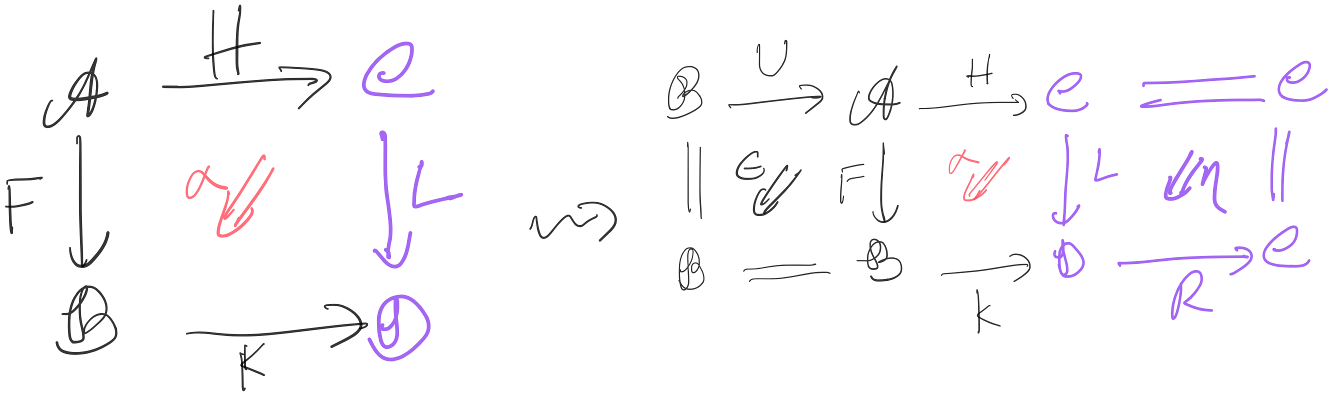
Consider the pair of double categories $\mathbb{L}\mathrm{adj}$ *and* $\mathbb{R}\mathrm{adj}$ *whose:*

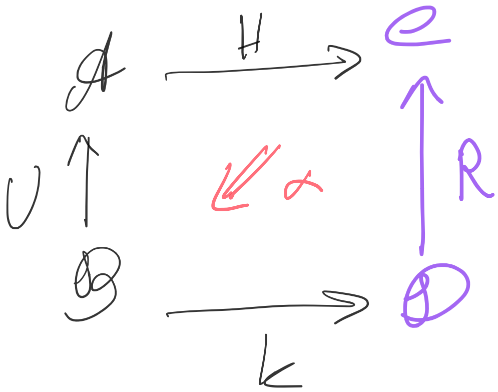
- *objects are categories,*
- *horizontal arrows are functors,*
- *vertical arrows are fully-specified adjunctions pointing in the direction of the left adjoint, and*
- *squares of* $\mathbb{L}\mathrm{adj}$ *(resp.* $\mathbb{R}\mathrm{adj}$*) are natural transformations between the squares of functors formed by the left (resp. right) adjoints.*

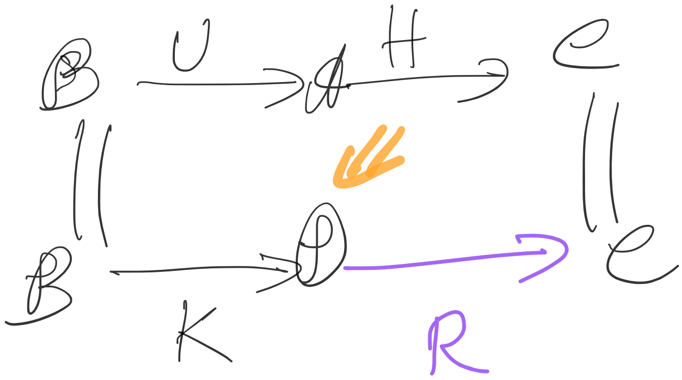
Then

$$\mathbb{L}\mathrm{adj} \cong \mathbb{R}\mathrm{adj}$$

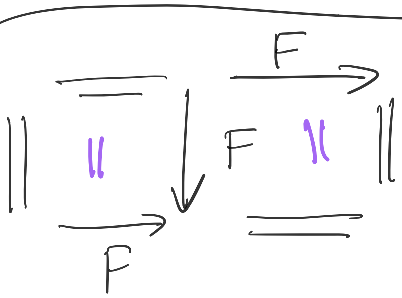
which acts on squares by taking mates.

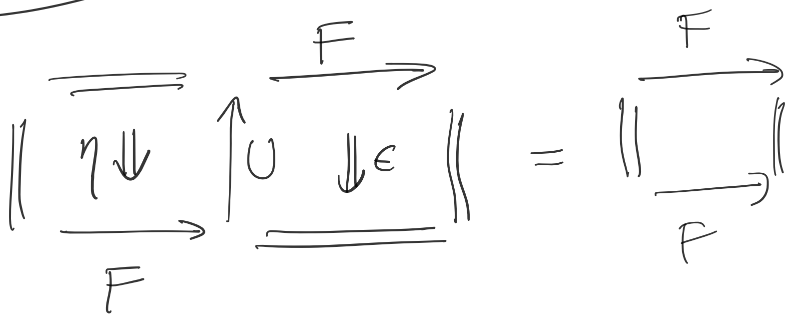






Unit & Counit As Mates





The Basic 2-cells

From the counit 2-cells

$$
\begin{array}{ccc}
/\Box & \xrightarrow{\Box_!} & /1 \\
{\scriptstyle \Box^*}\uparrow & \Downarrow \pi & \| \\
/1 & = & /1
\end{array}
\qquad\qquad
\begin{array}{ccc}
/1 & \xrightarrow{\Box^*} & /\Box \\
{\scriptstyle \Box_*}\uparrow & \Downarrow \nu & \| \\
/\Box & = & /\Box
\end{array}
$$

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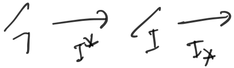

$$
\begin{array}{ccc}
/\square & \xrightarrow{\square_!} & /1 \\
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\end{array}
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\begin{array}{ccc}
/1 & \xrightarrow{\square^*} & /\square \\
{\scriptstyle \square_*}\uparrow & \Downarrow \nu & \| \\
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\end{array}
$$

we obtain the span

$$X^{\square} \xleftarrow{\pi} X^{\square} \times \square \xrightarrow{\epsilon} X$$

natural in X.

The Basic 2-cells

From the counit 2-cells

$$\begin{array}{ccc} /\Box & \xrightarrow{\Box_!} & /1 \\ {\scriptstyle \Box^*}\uparrow & \Downarrow \pi & \| \\ /1 & = & /1 \end{array} \qquad \begin{array}{ccc} /1 & \xrightarrow{\Box^*} & /\Box \\ {\scriptstyle \Box_*}\uparrow & \Downarrow \nu & \| \\ /\Box & = & /\Box \end{array}$$

we obtain the span

$$X^{\Box} \xleftarrow{\pi} X^{\Box} \times \Box \xrightarrow{\epsilon} X$$

natural in X.

Moreover, π is cartesian:

$$\begin{array}{ccc} A^{\Box} \times \Box & \xrightarrow{\pi} & A^{\Box} \\ {\scriptstyle p^{\Box} \times \Box}\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\Box}} \\ X^{\Box} \times \Box & \xrightarrow[\pi]{} & X^{\Box} \end{array}$$

Leibniz Exponential from the Basic 2-cells

The component of the whiskered counit

$$\begin{array}{ccccc} & & /X^{\square} & \xrightarrow{\pi^*} & /X^{\square} \times \square \\ & & {\scriptstyle \pi_*}\uparrow & \Downarrow \nu & \| \\ /X & \xrightarrow{\epsilon^*} & /X^{\square} \times \square & = & /X^{\square} \times \square \end{array}$$

at $p\colon A \to X$ is the Leibniz exponential $\delta \Rightarrow p\colon A^{\square} \times \square \to A_{\epsilon}$.

Constructing κ_ϵ via Mates

$$
\begin{array}{ccc}
A^{\square} \times \square & \xrightarrow{\ \epsilon\ } & A \\
{\scriptstyle p^{\square} \times \square}\downarrow & & \downarrow{\scriptstyle p} \\
X^{\square} \times \square & \xrightarrow[\ \epsilon\ ]{} & X
\end{array}
$$

Constructing κ_ϵ via Mates

$$
\begin{array}{ccc}
/A^{\square} \times \square & \xrightarrow{\epsilon_!} & /A \\
{\scriptstyle (p^{\square} \times \square)_!} \downarrow & & \downarrow {\scriptstyle p_!} \\
/X^{\square} \times \square & \xrightarrow[\epsilon_!]{} & /X
\end{array}
$$

Constructing κ_ϵ via Mates

$$
\begin{array}{ccc}
/A^{\square} \times \square & \xrightarrow{\ \epsilon_!\ } & /A \\
{\scriptstyle (p^{\square} \times \square)^*} \uparrow & \Downarrow & \uparrow {\scriptstyle p^*} \\
/X^{\square} \times \square & \xrightarrow[\ \epsilon_!\ ]{} & /X
\end{array}
$$

Constructing κ_ϵ via Mates

$$\begin{array}{ccc} /A^{\square} \times \square & \xleftarrow{\epsilon^*} & /A \\ {\scriptstyle (p^{\square}\times\square)^*}\uparrow & \cong & \uparrow{\scriptstyle p^*} \\ /X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X \end{array}$$

Constructing κ_ϵ via Mates

$$
\begin{array}{ccc}
/A^{\Box} \times \Box & \xleftarrow{\ \epsilon^*\ } & /A \\
{\scriptstyle (p^{\Box} \times \Box)_*} \big\downarrow & \Uparrow \kappa_\epsilon & \big\downarrow {\scriptstyle p_*} \\
/X^{\Box} \times \Box & \xleftarrow[\ \epsilon^*\ ]{} & /X
\end{array}
$$

Constructing κ_ϵ via Mates

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{\scriptstyle p^{\square}\times\square}\downarrow & & \downarrow{\scriptstyle p} \\
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/X^{\square} \times \square & \xrightarrow[\ \epsilon_!\ ]{} & /X
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\begin{array}{ccc}
/A^{\square} \times \square & \xleftarrow{\ \epsilon^*\ } & /A \\
{\scriptstyle (p^{\square}\times\square)_*}\downarrow & \Uparrow \kappa_\epsilon & \downarrow{\scriptstyle p_*} \\
/X^{\square} \times \square & \xleftarrow[\ \epsilon^*\ ]{} & /X
\end{array}
$$

The component of κ_ϵ at $q\colon B \to A$ defines a map $\kappa_\epsilon\colon (\Pi_A B)_\epsilon \to \Pi_{A^{\square}\times\square} B_\epsilon$ over $X^{\square} \times \square$.

Constructing κ_ϵ via Mates

So far,

$$\begin{array}{ccc}
(\Pi_A B)^{\Box} \times \Box & & \Pi_{A^{\Box} \times \Box} B^{\Box} \times \Box \\
{\scriptstyle \delta \Rightarrow p_* q}\downarrow & & \downarrow{\scriptstyle (p^{\Box} \times \Box)_* (\delta \Rightarrow q)} \\
(\Pi_A B)_\epsilon & \xrightarrow[\kappa_\epsilon]{} & \Pi_{A^{\Box} \times \Box}(B_\epsilon)
\end{array}$$

Constructing κ_ϵ via Mates

So far,

$$\begin{array}{ccc}
(\Pi_A B)^{\square} \times \square & \overset{\kappa}{\dashrightarrow} & \Pi_{A^{\square} \times \square} B^{\square} \times \square \\
{\scriptstyle \delta \Rightarrow p_* q} \downarrow & & \downarrow {\scriptstyle (p^{\square} \times \square)_* (\delta \Rightarrow q)} \\
(\Pi_A B)_\epsilon & \underset{\kappa_\epsilon}{\longrightarrow} & \Pi_{A^{\square} \times \square} (B_\epsilon)
\end{array}$$

Next, we find the top arrow.

Constructing κ from κ_ϵ

$$
\begin{array}{ccccccc}
/A^{\square} \times \square & \xleftarrow{\pi^*} & /A^{\square} & \xleftarrow{\pi_*} & /A^{\square} \times \square & \xleftarrow{\epsilon^*} & /A \\
{\scriptstyle (p^{\square} \times \square)_*}\big\downarrow & \cong & {\scriptstyle p^{\square}_*}\big\downarrow & \cong & \big\downarrow{\scriptstyle (p^{\square} \times \square)_*} & \Uparrow \kappa_\epsilon & \big\downarrow{\scriptstyle p_*} \\
/X^{\square} \times \square & \xleftarrow[\pi^*]{} & /X^{\square} & \xleftarrow[\pi_*]{} & /X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X
\end{array}
$$

Constructing κ from κ_ϵ

The component of this composite 2-cell at $q\colon B \to A$ defines a map

$$\kappa\colon (\Pi_A B)^{\square} \times \square \to \Pi_{A^{\square} \times \square}(B^{\square} \times \square)$$

over $X^{\square} \times \square$.

Constructing κ from κ_ϵ

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$$\kappa\colon (\Pi_A B)^{\square} \times \square \to \Pi_{A^{\square}\times\square}(B^{\square} \times \square)$$

over $X^{\square} \times \square$.

$$\begin{array}{ccccccc}
B^{\square} \times \square / A^{\square} \times \square & \xleftarrow{\ \pi^*\ } & B^{\square}/A^{\square} & \xleftarrow{\ \pi_*\ } & B_\epsilon / A^{\square} \times \square & \xleftarrow{\ \epsilon^*\ } & B/A \\
\downarrow{\scriptstyle (p^{\square}\times\square)_*} & & & & & & \downarrow{\scriptstyle p_*} \\
(\Pi_A B)^{\square} \times \square / X^{\square} \times \square & \xleftarrow[\ \pi^*\ ]{} & (\Pi_A B)^{\square}/X^{\square} & \xleftarrow[\ \pi_*\ ]{} & (\Pi_A B)_\epsilon / X^{\square} \times \square & \xleftarrow[\ \epsilon^*\ ]{} & \Pi_A B / X
\end{array}$$

Constructing κ from κ_ϵ

We now have to verify that the square below commutes.

$$
\begin{array}{ccc}
(\Pi_A B)^{\square} \times \square & \xrightarrow{\kappa} & \Pi_{A^{\square} \times \square} B^{\square} \times \square \\
{\scriptstyle \delta \Rightarrow p_* q} \Big\downarrow & & \Big\downarrow {\scriptstyle (p^{\square} \times \square)_*(\delta \Rightarrow q)} \\
(\Pi_A B)_\epsilon & \xrightarrow[\kappa_\epsilon]{} & \Pi_{A^{\square} \times \square}(B_\epsilon)
\end{array}
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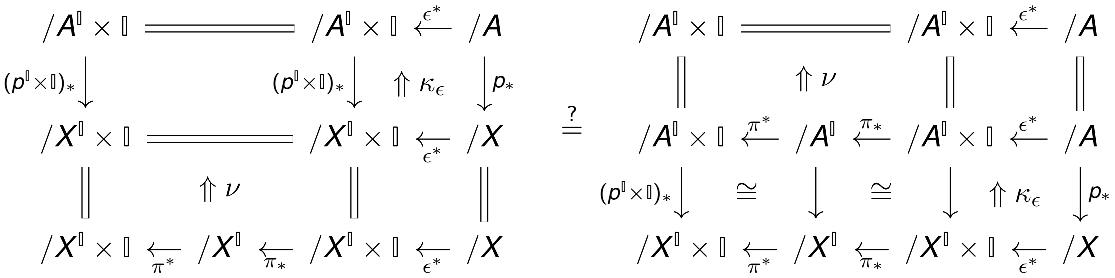
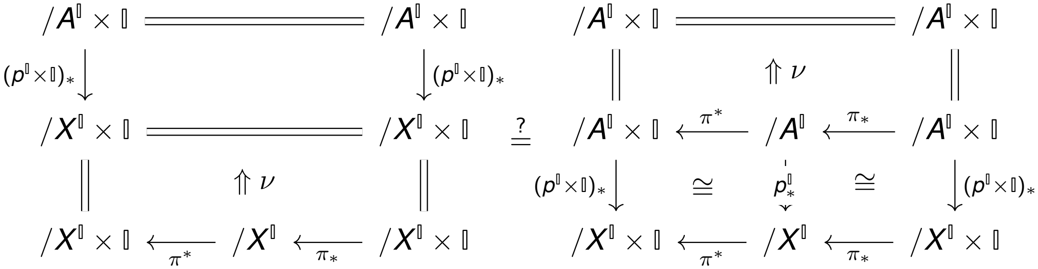
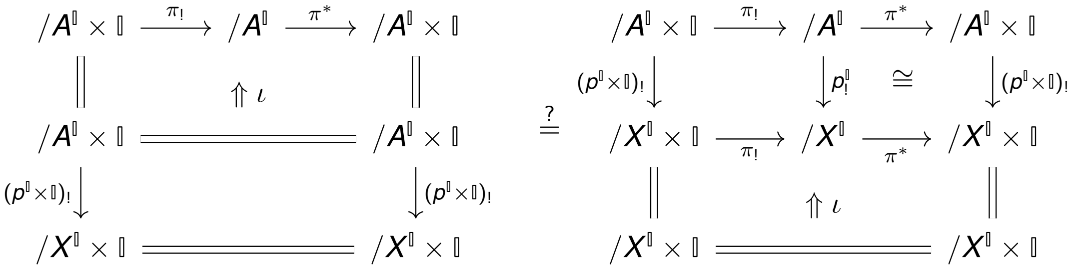
Constructing κ from κ_ϵ



Constructing κ from κ_ϵ



Constructing κ from κ_ϵ



Constructing κ from κ_ϵ

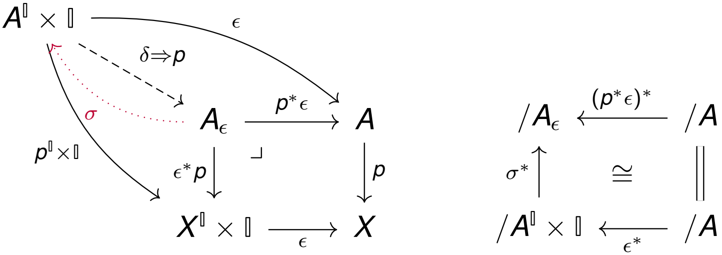
$$
\begin{array}{ccccc}
/A^{\square} \times \square & \xrightarrow{\pi_!} & /A^{\square} & \xrightarrow{p^{\square}_!} & /X^{\square} \\
\Vert & \Uparrow \iota & \downarrow \pi^* & \cong & \downarrow \pi^* \\
/A^{\square} \times \square & = & /A^{\square} \times \square & \xrightarrow[(p^{\square} \times \square)_!]{} & /X^{\square} \times \square
\end{array}
\overset{?}{=}
\begin{array}{ccccc}
/A^{\square} \times \square & \xrightarrow{(p^{\square} \times \square)_!} & /X^{\square} \times \square & \xrightarrow{\pi_!} & /X^{\square} \\
\Vert & & \Vert & \Uparrow \iota & \downarrow \pi^* \\
/A^{\square} \times \square & \xrightarrow[(p^{\square} \times \square)_!]{} & /X^{\square} \times \square & = & /X^{\square} \times \square
\end{array}
$$

Constructing κ from κ_ϵ

$$
\begin{array}{ccccc}
/A^{\square} \times \square & \xrightarrow{\pi_!} & /A^{\square} & \xrightarrow{p^{\square}_!} & /X^{\square} \\
\Vert & & \uparrow{\scriptstyle \pi_!} & & \uparrow{\scriptstyle \pi_!} \\
/A^{\square} \times \square & = & /A^{\square} \times \square & \xrightarrow[(p^{\square}\times\square)_!]{} & /X^{\square} \times \square
\end{array}
\quad = \quad
\begin{array}{ccccc}
/A^{\square} \times \square & \xrightarrow{(p^{\square}\times\square)_!} & /X^{\square} \times \square & \xrightarrow{\pi_!} & /X^{\square} \\
\Vert & & \Vert & & \uparrow{\scriptstyle \pi_!} \\
/A^{\square} \times \square & \xrightarrow[(p^{\square}\times\square)_!]{} & /X^{\square} \times \square & = & /X^{\square} \times \square
\end{array}
$$

Constructing the Retract ρ_ϵ

Since p is a fibration, $\delta \Rightarrow p$ is a trivial fibration and therefore it has a section:



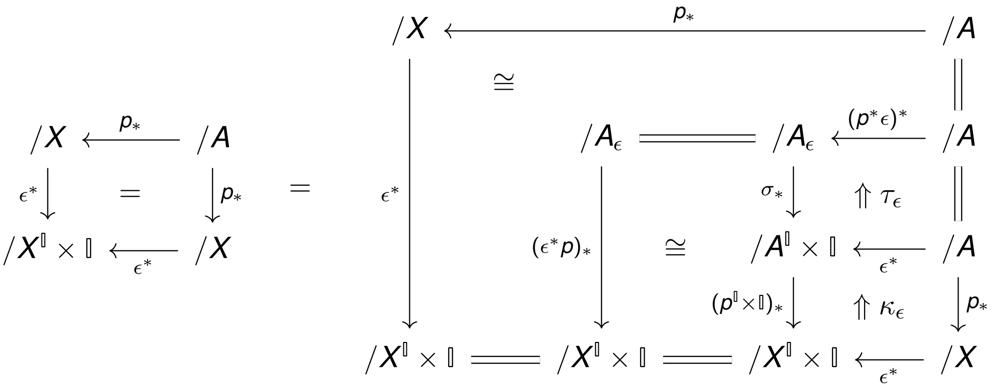
Constructing the Retract ρ_ϵ

$$
\begin{array}{ccc}
/A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
{\scriptstyle (\epsilon^* p)^*}\uparrow & \cong & \uparrow{\scriptstyle p^*} \\
/X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X
\end{array}
\quad = \quad
\begin{array}{ccccc}
/A_\epsilon & = & /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 & & {\scriptstyle \sigma^*}\uparrow & \cong & \| \\
{\scriptstyle (\epsilon^* p)^*}\uparrow & \cong & /A^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /A \\
 & & {\scriptstyle (p^{\square}\times\square)^*}\uparrow & \cong & \uparrow{\scriptstyle p^*} \\
/X^{\square} \times \square & = & /X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X
\end{array}
$$

Constructing the Retract ρ_ϵ

$$
\begin{array}{ccc}
/A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
{\scriptstyle (\epsilon^* p)_*}\downarrow & \cong & \downarrow{\scriptstyle p_*} \\
/X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X
\end{array}
\quad = \quad
\begin{array}{ccccc}
/A_\epsilon & = & /A_\epsilon & \xleftarrow{(p^*\epsilon)^*} & /A \\
 & & {\scriptstyle \sigma_*}\downarrow & \Uparrow \tau & \| \\
{\scriptstyle (\epsilon^* p)_*}\downarrow & \cong & /A^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /A \\
 & & {\scriptstyle (p^{\square} \times \square)_*}\downarrow & \Uparrow \kappa_\epsilon & \downarrow{\scriptstyle p_*} \\
/X^{\square} \times \square & = & /X^{\square} \times \square & \xleftarrow[\epsilon^*]{} & /X
\end{array}
$$

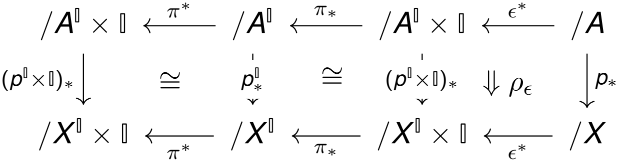
Constructing the Retract ρ_ϵ



Constructing the Retract ρ_ϵ

$$
\begin{array}{ccccc}
(\Pi_A B)^{\mathbb{I}} \times \mathbb{I} & \xrightarrow{\kappa} & \Pi_{A^{\mathbb{I}} \times \mathbb{I}} B^{\mathbb{I}} \times \mathbb{I} & & (\Pi_A B)^{\mathbb{I}} \times \mathbb{I} \\
{\scriptstyle \delta \Rightarrow p_* q}\big\downarrow & & \big\downarrow{\scriptstyle (p^{\mathbb{I}} \times \mathbb{I})_*(\delta \Rightarrow q)} & & \big\downarrow{\scriptstyle \delta \Rightarrow p_* q} \\
(\Pi_A B)_\epsilon & \xrightarrow[\kappa_\epsilon]{} & \Pi_{A^{\mathbb{I}} \times \mathbb{I}} (B_\epsilon) & \xrightarrow[\rho_\epsilon]{} & (\Pi_A B)_\epsilon
\end{array}
$$

Constructing ρ from ρ_ϵ



Completing the Proof

Similar to the commutativity of the square involving κ_ϵ and κ we show that the following square commutes:

$$\begin{array}{ccc}
\Pi_{A^{\square}\times\square} B^{\square} \times \square & \xrightarrow{\ \rho\ } & (\Pi_A B)^{\square} \times \square \\
{\scriptstyle (p^{\square}\times\square)_*(\delta\Rightarrow q)}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta\Rightarrow p_* q} \\
\Pi_{A^{\square}\times\square}(B_\epsilon) & \xrightarrow[\ \rho_\epsilon\ ]{} & (\Pi_A B)_\epsilon
\end{array}$$

That ρ is a retract of κ follows from the fact that ρ_ϵ is a retract of κ_ϵ and the iso 2-cells pasted to the left of κ_ϵ and ρ_ϵ, respectively, are pairwise inverses.

$$\begin{array}{ccccc}
(\Pi_A B)^{\square} \times \square & \xrightarrow{\ \kappa\ } & \Pi_{A^{\square}\times\square} B^{\square} \times \square & \xrightarrow{\ \rho\ } & (\Pi_A B)^{\square} \times \square \\
{\scriptstyle \delta\Rightarrow p_* q}\Big\downarrow & & \Big\downarrow{\scriptstyle (p^{\square}\times\square)_*(\delta\Rightarrow q)} & & \Big\downarrow{\scriptstyle \delta\Rightarrow p_* q} \\
(\Pi_A B)_\epsilon & \xrightarrow[\ \kappa_\epsilon\ ]{} & \Pi_{A^{\square}\times\square}(B_\epsilon) & \xrightarrow[\ \rho_\epsilon\ ]{} & (\Pi_A B)_\epsilon
\end{array}$$