Conservativity of

The Calculus of Constructions

over

Higher-order Heyting Arithmetic

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Overview

We investigate the relation between arithmetic and type theory.

We compare:

- Higher-order Heyting Arithmetic (HAH), and
- The Calculus of Constructions (CC),
  along with additional assumptions (CC$+$).
  $\underbrace{}_{\mathbb{N}, \Sigma, \mathrm{W}, \text{propext}, \text{funext}}$

We will show that CC$+$ is a conservative extension of HAH.

Contents

arithmetic $\rightleftarrows$ type theory

Higher-order Heyting Arithmetic

In higher-order logic we can quantify over powersets of the domain.
If we write $\exists x^n$ or $\forall x^n$ then x is an element of the n-th powerset:

- x^0 is an element of the domain,
- x^1 is a set,
- x^2 is a set of sets,
- and so on.

For x^n and Y^{n+1} we have a new atomic formula $x \in Y$.
We have two additional logical axiom schemes:

$$\forall X, Y^{n+1} \, (\forall z^n \, (z \in X \leftrightarrow z \in Y) \rightarrow X = Y), \quad \text{(extensionality)}$$

$$\exists X^{n+1} \, \forall z^n \, (z \in X \leftrightarrow P[z]). \quad \text{(comprehension)}$$

HAH has the axioms of PA but in intuitionistic higher-order logic.

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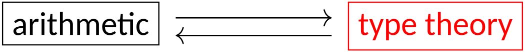

The Calculus of Constructions

CC is a minimalistic and impredicative version of type theory.

There are only two primitive types: Type_0 and Type_1.

We view these as universes and we assume $\mathsf{Type}_0 : \mathsf{Type}_1$.

We have only one way to construct new types:

$$\frac{A : \mathsf{Type}_i \qquad x : A \vdash B[x] : \mathsf{Type}_j}{\Pi(x : A)\, B[x] : \mathsf{Type}_j}\ (\Pi\text{-F, impredicative}),$$

Terms of $\Pi(x : A)\, B[x]$ are functions: they map $x : A$ to $y : B[x]$.

We write $A \to B$ for $\Pi(x : A)\, B$.

Compare this rule to Martin-Löf Type Theory where we have:

$$\frac{A : \mathsf{Type}_i \qquad x : A \vdash B[x] : \mathsf{Type}_j}{\Pi(x : A)\, B[x] : \mathsf{Type}_{\max\{i,j\}}}\ (\Pi\text{-F, predicative}),$$

Dependent Functions

Examples of types are:

$$\Pi(X : \mathsf{Type}_0)\,(X \to X) : \mathsf{Type}_0,$$

$$\mathsf{Type}_0 \to \mathsf{Type}_0 : \mathsf{Type}_1.$$

We can define functions and apply them:

$$\frac{\Pi(x : A)\,B[x] : \mathsf{Type}_i \qquad x : A \vdash b[x] : B[x]}{\lambda(x : A)\,b[x] : \Pi(x : A)\,B[x]}\ (\Pi\text{-I}),$$

$$\frac{f : \Pi(x : A)\,B[x] \qquad a : A}{f\,a : B[a]}\ (\Pi\text{-E}),$$

We can define for example:

$$\mathsf{id} :\ \ \Pi(X : \mathsf{Type}_0)\,(X \to X),$$

$$\mathsf{id} := \lambda(X : \mathsf{Type}_0)\,\lambda(x : X)\,x.$$

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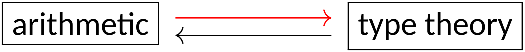

Higher-order Logic in The Calculus of Constructions

Think of $A : \mathsf{Type}_0$ as a proposition and of $a : A$ as a proof for A.

We write $\forall(x : A)\, B[x]$ for $\Pi(x : A)\, B[x]$ if we have $B[x] : \mathsf{Type}_0$.

The other logical connectives can be defined:

$$
\begin{aligned}
\bot &:= \forall(C : \mathsf{Type}_0)\, C, \\
\top &:= \forall(C : \mathsf{Type}_0)\, (C \to C), \\
A \vee B &:= \forall(C : \mathsf{Type}_0)\, ((A \to C) \to ((B \to C) \to C)), \\
A \wedge B &:= \forall(C : \mathsf{Type}_0)\, ((A \to (B \to C)) \to C), \\
\exists(x : A)\, B[x] &:= \forall(C : \mathsf{Type}_0)\, (\forall(x : A)\, (B[x] \to C) \to C), \\
\mathcal{P}\, A &:= A \to \mathsf{Type}_0, \\
(a =_A a') &:= \forall(P : \mathcal{P}\, A)\, (P a \to P a').
\end{aligned}
$$

Natural Numbers

We can define a weak version of $\mathbb{N}$:

$$\mathbb{N}_{\mathsf{w}} : \mathsf{Type}_0,$$
$$\mathbb{N}_{\mathsf{w}} := \Pi(Z : \mathsf{Type}_0)\,(Z \to ((Z \to Z) \to Z)).$$

The idea is to encode n as $\lambda Z\,\lambda z\,\lambda f\,f^n\,z$. We can define 0 and S:

$$0 : \mathbb{N}_{\mathsf{w}},$$
$$0 := \lambda(Z : \mathsf{Type}_0)\,\lambda(z : Z)\,\lambda(f : Z \to Z)\,z,$$

$$\mathsf{S} : \mathbb{N}_{\mathsf{w}} \to \mathbb{N}_{\mathsf{w}},$$
$$\mathsf{S} := \lambda(n : \mathbb{N}_{\mathsf{w}})\,\lambda(Z : \mathsf{Type}_0)\,\lambda(z : Z)\,\lambda(f : Z \to Z)\,f\,(n\,Z\,z\,f).$$

Natural Numbers

$\mathbb{N}_{\mathsf{w}}$ satisfies the rule:

$$\frac{C : \mathsf{Type}_0 \qquad c : C \qquad f : C \to C}{\mathsf{rec}_{C,c,f} : \mathbb{N} \to C} \ (\mathbb{N}\text{-E, weak}),$$

Simply take $\mathsf{rec}_{C,c,f} := \lambda(n : \mathbb{N}_{\mathsf{w}})\, n\, C\, c\, f$.

However this is weaker than the following rule:

$$\frac{n : \mathbb{N} \vdash C[n] : \mathsf{Type}_i \quad c : C[0] \quad f : \Pi(n : \mathbb{N})\,(C[n] \to C[\mathsf{S}\, n])}{\mathsf{ind}_{C,c,f} : \Pi(n : \mathbb{N})\, C[n]} \ (\mathbb{N}\text{-E}),$$

We can not define a $\mathbb{N} : \mathsf{Type}_0$ satisfying $\mathbb{N}$-E in CC. (Geuvers, 2001)

So, we cannot prove induction in CC.

In addition, we cannot prove extensionality or $0 \neq 1$. (Smith, 1988)

Additional Assumptions

We replace $\text{Type}_0 : \text{Type}_1$ with Prop, Set : Type.

We assume that there exists a $\mathbb{N}$: Set satisfying $\mathbb{N}$-E.

We also add $\mathbb{0}$, $\mathbb{1}$, $A + B$, $\Sigma(x : A)\, B[x]$, $\mathrm{W}(x : A)\, B[x]$, and $\|A\|$.

This brings us closer to CIC, which is implemented by Coq and Lean.

Lastly, we assume two axioms:

$$\textsf{funext} : \forall(f, f' : \Pi(x : A)\, B[x])\, (\forall(x : A)\, (f\, x = f'\, x) \to f = f'),$$

$$\textsf{propext} : \forall(P, P' : \textsf{Prop})\, ((P \to P') \wedge (P' \to P) \to P = P').$$

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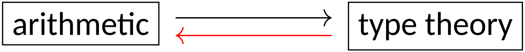


Main Result

Theorem

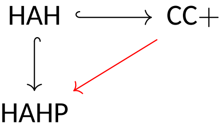
CC+ is a conservative extension of HAH.

Proof Sketch. We can show that CC+ proves the axioms of HAH. The difficult part is showing that it does not prove more.

We first give a conservative extension of HAH, named HAHP.

$\overbrace{\lambda x\, b[x], \{f\}\,(a), \langle a,b\rangle}$

Then we construct an arrow:



And show that the diagram commutes up to logical equivalence. □

Interpreting Propositions in HAHP

We will interpret the propositions, sets, and types of CC$+$ in HAHP.

Propositions are easy, we can interpret them as follows:

Definition (subsingleton)

A subsingleton is a set $P \subseteq \{0\}$.

A morphism from P to Q is just a function $P \to Q$.

Interpreting Sets in HAHP

Sets are more difficult because the type theory is impredicative.

We have to put restrictions on functions to avoid cardinality issues:

Definition (partial equivalence relation)

A PER is a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ that is symmetric and transitive.

We define:

$$\mathrm{dom}(R) := \{n \in \mathbb{N} \mid \langle n, n \rangle \in R\}, \qquad \text{(domain)}$$

$$[n]_R := \{m \in \mathbb{N} \mid \langle n, m \rangle \in R\}, \qquad \text{(equivalence class)}$$

$$\mathbb{N}/R := \{[n]_R \mid n \in \mathrm{dom}(R)\}. \qquad \text{(quotient)}$$

A morphism from R to S is a function $F : \mathbb{N}/R \to \mathbb{N}/S$ such that there exists a computable $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that:

$$n \in \mathrm{dom}(R) \text{ implies } f(n) \in F([n]_R).$$

Interpreting Types in HAHP

We interpret types in a similar way:

Definition (assembly)

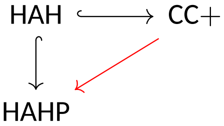
An assembly consists of an $A \subseteq \mathcal{P}^n(\mathbb{N})$ and a relation $\Vdash_A \subseteq \mathbb{N} \times A$ such that for every $a \in A$ there exists an $n \in \mathbb{N}$ with $n \Vdash_A a$.
A morphism from A to B is a function $F : A \to B$ such that there exists a computable $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that:

$$n \Vdash_A a \text{ implies } f(n) \Vdash_{\mathcal{B}} F(A).$$

Conservativity

This gives us a model of CC$+$ and an interpretation of CC$+$ in HAHP.

The following diagram is commutative (up to logical equivalence):



We conclude:

CC$+$ is a conservative extension of HAH,

λP$2+$ is a conservative extension of HA2,

λP$+$ is a conservative extension of HA.

Martin-Löf Type Theory

ML is not impredicative so our logical definitions do not work.

However, we can interpret higher-order logic as follows:

$$\bot^* := \mathbb{0}, \qquad (a^n \in X^{n+1})^* := X\,a,$$

$$\top^* := \mathbb{1}, \qquad (a^n = b^n)^* := (a =_{\mathcal{P}^n \mathbb{N}} b),$$

$$(A \vee B)^* := A^* + B^*, \qquad (\exists x^n\, B(x^n))^* := \Sigma(x : \mathcal{P}^n\, \mathbb{N})\, B(x^n)^*,$$

$$(A \wedge B)^* := A^* \times B^*, \qquad (\forall x^n\, B(x^n))^* := \Pi(x : \mathcal{P}^n\, \mathbb{N})\, B(x^n)^*,$$

$$(A \to B)^* := A^* \to B^*.$$

For this interpretation, ML1 is not conservative over HA2:

ML1 proves choice but not extensionality or comprehension.

Martin-Löf Type Theory

Alternatively, with $\|\cdot\|$ we can interpret higher-order logic as follows:

$$\bot^\circ := \mathbb{0}, \qquad (a^n \in X^{n+1})^\circ := \|X\,a\|,$$

$$\top^\circ := \mathbb{1}, \qquad (a^n = b^n)^\circ := (a =_{\mathcal{P}^n \mathbb{N}} b),$$

$$(A \vee B)^\circ := \|A^\circ + B^\circ\|, \quad (\exists x^n\, B(x^n))^\circ := \|\Sigma(x : \mathcal{P}^n\,\mathbb{N})\, B(x^n)^\circ\|,$$

$$(A \wedge B)^\circ := A^\circ \times B^\circ, \quad (\forall x^n\, B(x^n))^\circ := \Pi(x : \mathcal{P}^n\,\mathbb{N})\, B(x^n)^\circ,$$

$$(A \to B)^\circ := A^\circ \to B^\circ.$$

For this interpretation, ML1 with $\|A\| : \mathrm{Type}_0$ might be conservative over HA2 without extensionality.

Summary

For impredicative type theory we have:

$$\text{CC}+ \text{ is a conservative extension of HAH,}$$

$$\lambda\text{P2}+ \text{ is a conservative extension of HA2,}$$

$$\lambda\text{P}+ \text{ is a conservative extension of HA,}$$

For predicative type theory we have:

ML1 is not a conservative extension of HA2 using $*$,

$\text{ML1} + \| \cdot \|$ is a conservative extension of $\text{HA2} - \text{ext}$ using $\circ$.

The last result is still work in progress.