Symmetric effective Kan complexes with pictures

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supervised by Benno van den Berg

January 16, 2023

1. Kan complexes
    - The simplex category
    - Simplicial sets
    - Models for homotopy type theory

2. Symmetric effective Kan complexes
    - Pictures
    - Diagrams
    - Definition

The simplex category

Define the simplex category Δ as follows:

- Objects are of the form $[n] = \mathbb{N}_{\leq n}$. We see them as linearly ordered sets of size $n + 1$.
- Morphisms are order-preserving functions.

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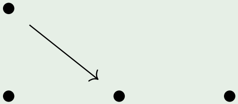
Example



(1)

is a morphism in Δ from [0] to [2]

Degeneracy maps

There are two special classes of morphisms in Δ.

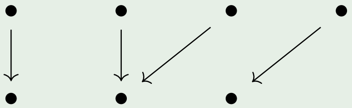
- For $0 \leq i \leq n$, we have a degeneracy map

$$s_i : [n+1] \to [n] \tag{2}$$

hitting i twice.

$$s_i(k) = \begin{cases} k \text{ if } k \leq i \\ k-1 \text{ if } k > i \end{cases} \tag{3}$$

Example



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Face maps

There are two special classes of morphisms in Δ.

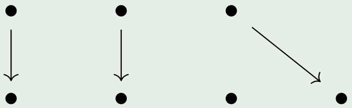
- For $0 \leq i \leq n$, we have a face map

$$d_i : [n] \to [n+1] \tag{5}$$

skipping over i.

$$d_i(k) = \begin{cases} k \text{ if } k < i \\ k+1 \text{ if } k \geq i \end{cases} \tag{6}$$

Example



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Some Remarks

- All morphisms in Δ can be written as $m \circ e$, where e is a composition of degeneracy maps and m a composition of face maps.
- There are some composition laws:

$$s_j \circ d_k = \begin{cases} d_{k-1} \circ s_j \text{ if } k > j+1 \\ 1 \text{ if } k \in \{j, j+1\} \\ d_k \circ s_{j-1} \text{ if } k < j \end{cases} \tag{8}$$

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In general, we will write $s_j \circ d_k = d_{k'} \circ s_{j'}$ if $k \neq j, j+1$.

Simplicial Sets

A simplicial set is a presheaf on Δ.

More generally, a simplicial object in a category $\mathcal{C}$ is a functor

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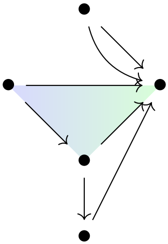
Because morphisms in Δ are generated by face and degeneracy maps, to give a simplicial set X, we need to give

- for each $n \in \mathbb{N}$ a set X_n.
- An action on degeneracy maps $X(s_i) : X_n \to X_{n+1}$.
- An action on face maps $X(d_i) : X_{n+1} \to X_n$.

Example: Oriented hypergraphs



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The standard simplex

Recall that we have for each object $[n]$ of Δ a simplicial set of morphisms into $[n]$.

$$\Delta^n = \Delta\left[(-), [n]\right] \tag{11}$$

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- There is a lattice structure on subobjects of the standard simplex (sieves).
- Presheaves have an internal logic.

Horns

- For all $i \leq n$, we have all faces of the standard simplex $d_i \subseteq \Delta^n$, containing morphisms that factor trough d_i.

$$d_i([m]) = \{f : [m] \to [n] | f \text{ doesn't hit } i\} \tag{12}$$

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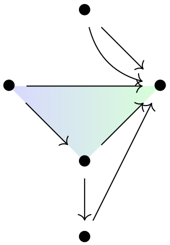
$$d_i([m]) = \{f : [m] \to [n] | f \text{ doesn't hit } i\} \tag{12}$$

- We also have horns Λ^n_k for all $0 \leq k \leq n$.

$$\Lambda^n_k = \bigcup_{i \neq k} d_i \tag{13}$$

If $k \neq 0, n$, we call Λ^n_k an inner horn.

Example: Oriented hypergraphs



(14)

G

(15)

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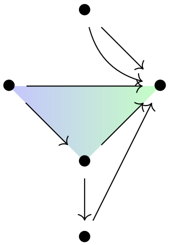

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$$\Delta^2 \longrightarrow G \quad (15)$$

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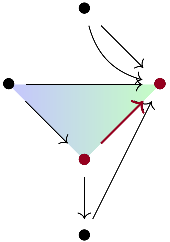

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$$\Delta^1 \xrightarrow{d_0} \Delta^2 \longrightarrow G$$

(15)

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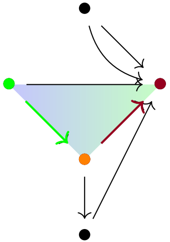

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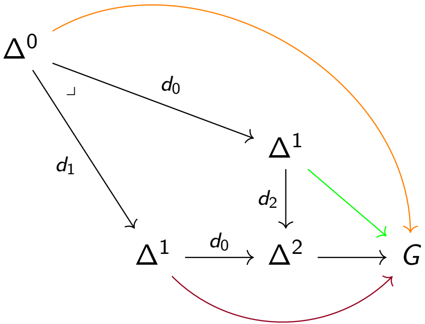

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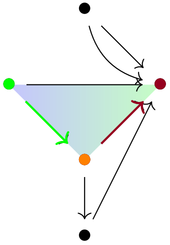

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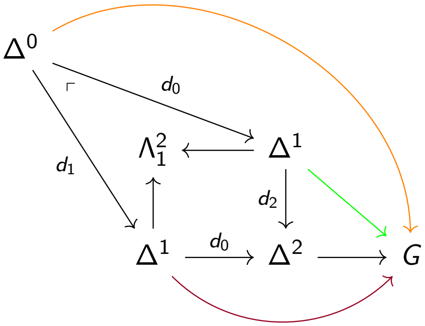

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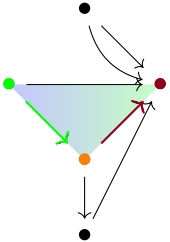

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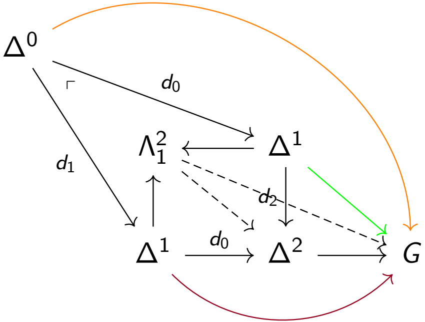

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∞-categories and Kan complexes

Let G be a simplicial set. Consider problems of the form

$$\begin{array}{ccc} \Lambda^n_m & \xrightarrow{y} & G \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \tag{16}$$

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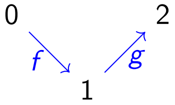
- Is a Kan complex (or ∞-groupoid) if it always has solutions for the above problem.

∞-categories and Kan complexes

- ∞-categories have a notion of composition.



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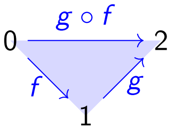

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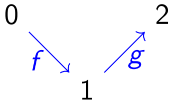

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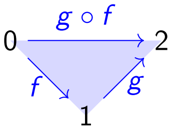

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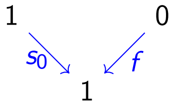
where degeneracy maps give identities.

- Kan complexes also have a notion of inverses.



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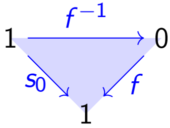

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Models for homotopy type theory

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- Dependent types are based on Kan fibrations.

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$$

- λ-abstraction corresponds to pushforward of Kan fibrations.
- Problem: the proof that Kan fibrations are closed under pushforwards is not constructive.

Criticism on Kan complexes

First of all, we are not satisfied with mere existence of fillers. We want functional fillers.

$$\begin{array}{ccc} \Lambda^n_m & \xrightarrow{y} & G \\ \big\downarrow & \nearrow_{\mathrm{fil}(y)} & \\ \Delta^n & & \end{array} \tag{22}$$

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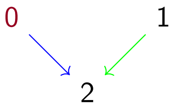
And we want this function fil to satisfy some structural properties. Benno and Eric reduced this property to stability under pullback along degeneracy maps.

Example of the condition

We start of by a Λ^2_2-horn.



(23)

$$\begin{array}{ccc} \Lambda^n_m & \xrightarrow{y} & G \\ \big\downarrow & & \\ \Delta^n & & \end{array} \tag{24}$$

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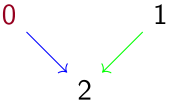

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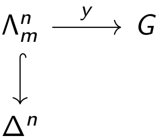
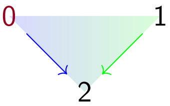


(24)

For which we have a filler:



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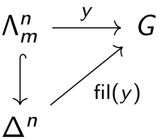

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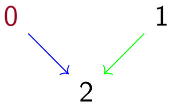
Pulling back a horn along a degeneracy map: the picture



(27)

We pull back our horn along s_0.

Pulling back a horn along a degeneracy map: the picture

$$\begin{array}{ccc} 0 & & 1 \\ & \searrow \quad \swarrow & \\ & 2 & \end{array} \tag{27}$$

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$$\begin{array}{ccccccc} 0 & \leq & 1 & \leq & 2 & \leq & 3 \\ \downarrow & \swarrow & & \swarrow & & \swarrow & \\ 0 & \leq & 1 & \leq & 2 & & \end{array} \tag{28}$$

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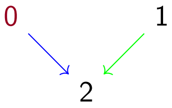

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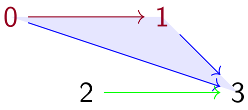
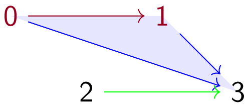
Geometrically, we stretch out the red point 0 to a line.



(29)

Getting a new horn



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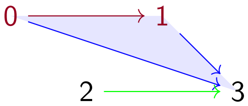
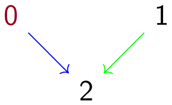


(30)

Notice: d_0 and d_1 are our original horn.



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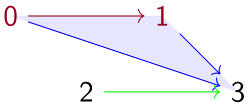

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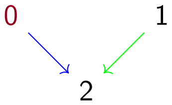
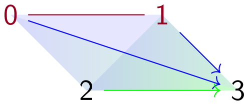


(31)

We can add our filler here:



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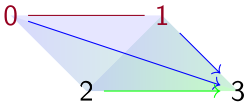
The condition for an effective Kan complex

We require that the chosen filler for:



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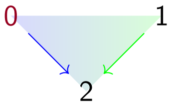
Is exactly



(34)

pulled back along s_0.

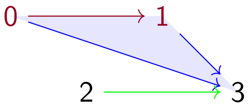
Pulling back a horn along a degeneracy map: the diagram

$$\begin{array}{ccc} \Lambda^n_m & \xrightarrow{y} & G \\ \big\downarrow & & \\ \Delta^n & & \end{array} \tag{35}$$

Pulling back a horn along a degeneracy map: the diagram

$$\begin{array}{ccccc} s_j^*(\Lambda^n_m) & \longrightarrow & \Lambda^n_m & \xrightarrow{y} & G \\ \big\downarrow & \lrcorner & \big\downarrow & & \\ \Delta^{n+1} & \xrightarrow{s_j} & \Delta^n & & \end{array} \tag{35}$$

We study $s_j^*(\Lambda^n_m) \subseteq \Delta^{n+1}$ "facewise".



(36)

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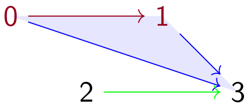
Recall that $s_j \circ d_k = d_{k'} \circ s_{j'}$ if $k \neq j, j+1$.



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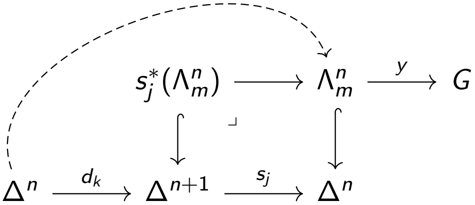

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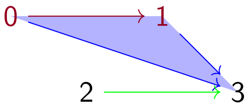

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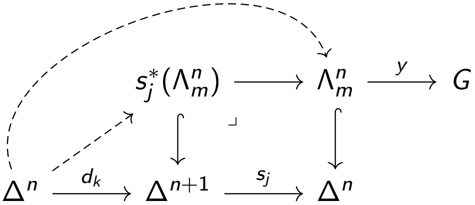
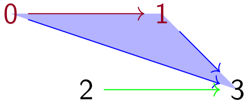
Recall that $s_j \circ d_k = d_{k'} \circ s_{j'}$ if $k \neq j, j+1$.
So if $k' \neq m$, this face factors trough $d_{k'} \subseteq \Lambda^n_m$.

Pulling back a horn along a degeneracy map: the diagram



(35)

We study $s_j^*(\Lambda^n_m) \subseteq \Delta^{n+1}$ "facewise".



(36)

Recall that $s_j \circ d_k = d_{k'} \circ s_{j'}$ if $k \neq j, j+1$. So if $k' \neq m$, this face factors trough $d_{k'} \subseteq \Lambda^n_m$. Hence we know the value of our map on this face. If $k' = m$, we set $m^* = k$, which will be our new missing face.

Pulling back a horn along a degeneracy map: the diagram

$$
\begin{array}{ccccccc}
 & & s_j^*(\Lambda^n_m) & \longrightarrow & \Lambda^n_m & \xrightarrow{y} & G \\
 & & \big\downarrow & \lrcorner & \big\downarrow & & \\
\Delta^n & \xrightarrow{d_j/d_{j+1}} & \Delta^{n+1} & \xrightarrow{s_j} & \Delta^n & &
\end{array}
\tag{35}
$$

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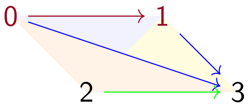

(36)

If $k \in \{j, j+1\}$, then $s_j \circ d_k = 1$.

Pulling back a horn along a degeneracy map: the diagram

$$
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\Lambda^n_m & \longrightarrow & s_j^*(\Lambda^n_m) & \longrightarrow & \Lambda^n_m & \xrightarrow{y} & G \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\
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\end{array}
\qquad (\Delta^n \xrightarrow{1} \Delta^n) \tag{35}
$$

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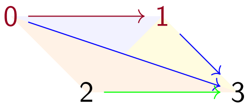

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If $k \in \{j, j+1\}$, then $s_j \circ d_k = 1$.
We recover exactly our original horn.

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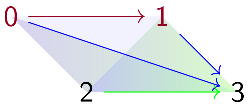
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \nearrow_{\mathrm{fil}(y)} & \\
\Delta^n & \xrightarrow{d_j/d_{j+1}} & \Delta^{n+1} & \xrightarrow{s_j} & \Delta^n & &
\end{array}
\qquad \text{with } s_j \circ (d_j/d_{j+1}) = 1 \tag{35}
$$

We study $s_j^*(\Lambda^n_m) \subseteq \Delta^{n+1}$ “facewise”.



(36)

If $k \in \{j, j+1\}$, then $s_j \circ d_k = 1$.
We recover exactly our original horn.
We have a chosen filler for these faces and get a new map $\Lambda^n_{m*} \to G$.

An extended pulled back horn in general

For any map $y : \Lambda^n_m \to G$, $s_j : \Delta^{n+1} \to \Delta^n$, we create a horn map $\Lambda^{n+1}_{m^*} \to G$ with

$$m^* \in \begin{cases} \{m\} \text{ if } m < j \\ \{m, m+1\} \text{ if } m = j \\ \{m+1\} \text{ if } m > j \end{cases} \tag{37}$$

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Define $s_j^*(y) : \Lambda^{n+1}_{m^*} \to G$ by

$$s_j^*(y) \circ d_k = \begin{cases} y \circ s_j \circ d_k \text{ if } k \neq j, j+1, m^* \\ \text{fil}(y) \text{ if } k \in \{j, j+1\} - \{m^*\} \end{cases} \tag{38}$$

A formal definition

A simplicial set G is an effective Kan complex if it comes equiped with an operation fil which

- Takes as input any horn map $y : \Lambda^n_m \to G$.
- Gives as output an extension $\text{fil}(y) : \Delta^n \to G$

In such a way that for any $0 \leq j \leq n$ and any $m^*, s_j^*(y)$ as described above, we have $\text{fil}(s_j^*(y)) = \text{fil}(y) \circ s_j$.

Effective Kan complexes

$$\begin{array}{ccccc} \partial(\Delta^n) & \longrightarrow & s_i^*(\partial(\Delta^n)) & \longrightarrow & \partial\Delta^n \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[d_i/d_{i+1}]{} & \Delta^{n+1} & \xrightarrow[s_i]{} & \Delta^n \end{array} \tag{39}$$

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The left square corresponds to

$$\bigcup_{k \neq i, i+1} d_k \cup (d_i/d_{i+1}) \tag{40}$$

Effective Kan complexes

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\end{array}
\tag{39}
$$

The left square corresponds to

$$\bigcup_{k\neq i,i+1} d_k \cup (d_i/d_{i+1}) \tag{40}$$

Every inner horn Λ^n_m can now be represented as

$$\bigcup_{k\neq m,m+1} d_k \cup d_{m+1}, \text{ or } \bigcup_{k\neq m-1,m} d_k \cup d_{m-1} \tag{41}$$

Simplicial Malcev algebras

Let A be the category of an algebraic theory. TFAE:

a. All simplicial objects of A are Kan complexes.
b. A allows for a Malcev operation.

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A Malcev operation $\mu(\cdot,\cdot,\cdot)$ satisfies

$$\mu(x,x,y) = y, \mu(x,y,y) = x \tag{42}$$

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Examples are:

- Groups, with $\mu(x,y,z) = xy^{-1}z$
- Heyting algebras with $\mu(x,y,z) = ((z \to y) \to x) \wedge ((x \to y) \to z)$.

Simplicial Malcev algebras

Let A be the category of an algebraic theory. TFAE:

a. All simplicial objects of A are Kan complexes.
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c. All simplicial objects of A are symmetric effective Kan complexes.

A Malcev operation $\mu(\cdot,\cdot,\cdot)$ satisfies

$$\mu(x,x,y) = y, \mu(x,y,y) = x \tag{42}$$

Examples are:

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Lifting against squares

$$
\begin{array}{ccc}
\bigcup\limits_{k\neq j,j+1,m^*} d_k & \longrightarrow & \Lambda^n_m \\
\downarrow & & \\
\bigcup\limits_{k\neq j,m^*} d_k & & \\
\downarrow & & \\
\bigcup\limits_{k\neq m^*} d_k & & \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \xrightarrow{s_j} & \Delta^n \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \xrightarrow{s_j} & \Delta^n
\end{array} \tag{43}
$$

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Note: on the left side we have a composition of pushouts of horn inclusions.

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\downarrow & & \Big\downarrow \\
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\downarrow & & \downarrow \\
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On the right as well.

Lifting against squares

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\end{array}
\qquad (43)
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If we explicitly save this information, Kan fibrations have lifts against such squares.

Lifting against squares

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\downarrow & & \Big| \\
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\downarrow & & \Big| \\
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Note: on the left side we have a composition of pushouts of horn inclusions.
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If we explicitly save this information, Kan fibrations have lifts against such squares.
This is a condition for a lifting AWFS.