Model-Theoretic Origin of Profinite Integers

DutchCATs

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Introduction

The Coherent Fragment of Categorical Logic

A *pretopos* $\mathcal{C}$ is a category that has finite limits, *universal* effective epimorphisms, and *universal* disjoint finite coproducts.

A *model* of a small pretopos $\mathcal{C}$ is a pretopos functor $\mathcal{C} \to \mathbf{Set}$.

- The initial pretopos is **FinSet**.
- $\mathbf{FinSet}^{\mathbb{Z}} := [\mathbb{Z}, \mathbf{FinSet}]$ is also a pretopos. Equivalently, it is the category of finite $\mathbb{Z}$-sets.

Question: What is $\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}})$?

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Question: What is $\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}})$?

Spoiler

We have the following result:

- $\mathbf{FinSet}^{\mathbb{Z}}$ is *categorical*: It has only 1 model upto isomorphism.
- However, the automorphism group of this model is non-trivial,

$$\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}}) \simeq \widehat{\mathbb{Z}},$$

  where $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ is the profinite completion of integers.
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Our strategy: Classify objects of $\mathbf{FinSet}^{\mathbb{Z}}$ upto finite limits and coproducts, because models $M : \mathbf{FinSet}^{\mathbb{Z}} \to \mathbf{Set}$ preserves them.

Well-known facts:

- All X are coproducts of transitive ones $X \cong \coprod_{[x]\in X/\sim}[x]$.
- Transitive systems are isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for some $m \geq 1$.
- For any $m, n \geq 1$,

$$\mathbf{FinSet}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & n \mid m \\ 0 & \text{otherwise} \end{cases}$$

- Composition of maps are modular addition

$$\begin{array}{ccc} \mathbb{Z}/m\mathbb{Z} & \xrightarrow{\ i\ } & \mathbb{Z}/n\mathbb{Z} \\ & \searrow^{i+j} & \downarrow j \\ & & \mathbb{Z}/k\mathbb{Z} \end{array}$$

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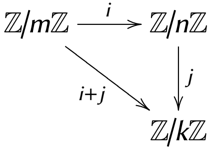

Structure of $\mathbf{FinSet}^{\mathbb{Z}}$

Decomposition upto products:

Lemma

If $\gcd(m, n) = 1$, *then* $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/mn\mathbb{Z}$.

More generally if $\gcd(m, n) = d$ *and* $\mathrm{lcm}(m, n) = k$,

$$[\langle 0, 0\rangle, \cdots, \langle 0, d-1\rangle] : \coprod_{d} \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Example

- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$.
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \sqcup \mathbb{Z}/2\mathbb{Z}$.

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Corollary

To determine the value of M, only need to care about the value of the following diagramme for all primes $p, q, \cdots$:

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{0} & \mathbb{Z}/p^k\mathbb{Z} \circlearrowright^{1} & \xrightarrow{0} & \mathbb{Z}/p^{k-1}\mathbb{Z} \circlearrowright^{1} & \xrightarrow{0} & \cdots & \xrightarrow{0} & \mathbb{Z}/p\mathbb{Z} \circlearrowright^{1} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & \xrightarrow{0} & \mathbb{Z}/q^k\mathbb{Z} \circlearrowright^{1} & \xrightarrow{0} & \mathbb{Z}/q^{k-1}\mathbb{Z} \circlearrowright^{1} & \xrightarrow{0} & \cdots & \xrightarrow{0} & \mathbb{Z}/q\mathbb{Z} \circlearrowright^{1}
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And different primes are independent from each other.

Models of FinSet$^{\mathbb{Z}}$

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Let $M : \mathbf{FinSet}^{\mathbb{Z}} \to \mathbf{Set}$ be a model:

- Let $M_{/p^k}$ to denote the value of $\mathbb{Z}/p^k\mathbb{Z}$ under M.
- $\mathrm{Aut}(\mathbb{Z}/p^k\mathbb{Z}) \cong \mathbb{Z}/p^k\mathbb{Z}$ induces an $\mathbb{Z}/p^k\mathbb{Z}$-action on $M_{/p^k}$.
- The isomorphism in $\mathbf{FinSet}^{\mathbb{Z}}$

$$[\langle 0, i\rangle_{1\le i<p^k}] : \coprod_{p^k} \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^k\mathbb{Z},$$

induces an isomorphism,

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These information suffices to determine $\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}})$ upto iso.

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Lemma

$M_{/p^k}$ is isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$.

Proof.

$$[\langle 0, i\rangle_{1\le i<p^k}] : \coprod_{p^k} M_{/p^k} \cong M_{/p^k} \times M_{/p^k}.$$

- Injectivity: For any $x \in M_{/p^k}$ and $i \neq j$, $x \cdot i \neq x \cdot j$.
  $\Rightarrow \mathbb{Z}/p^k\mathbb{Z}$-action on $M_{/p^k}$ is *free*.
- Surjectivity: For any $x, y \in M_{/p^k}$, there exists i that $x \cdot i = y$.
  $\Rightarrow \mathbb{Z}/p^k\mathbb{Z}$-action on $M_{/p^k}$ is *transitive*. □

Corollary

$\mathbf{FinSet}^{\mathbb{Z}}$ *is categorical.*

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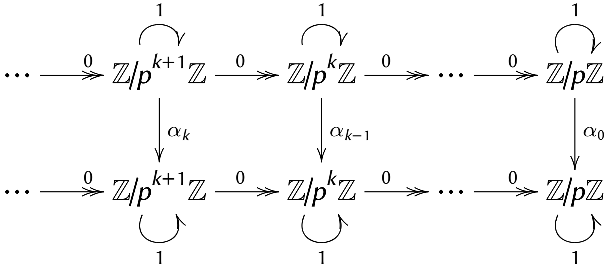

Observation

$\alpha = (\alpha_0, \alpha_1, \cdots)$ is natural (for prime p) iff $\alpha_k = \alpha_{k-1} \mod p^k$.
Equivalently, α is a p-adic integer $\alpha = \sum_{i=0}^{\infty} a_i p^i$, $\alpha_k = \sum_{i=0}^{k-1} a_i p^i$.

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$\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}}) \simeq \widehat{\mathbb{Z}} \cong \prod_{p\ prime} \mathbb{Z}_p$.

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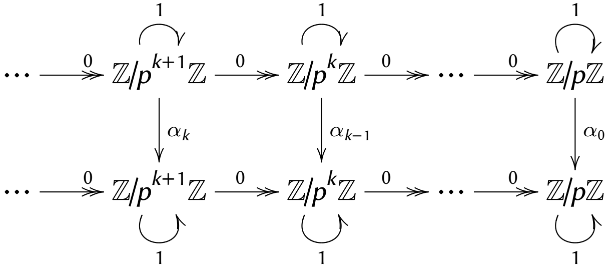

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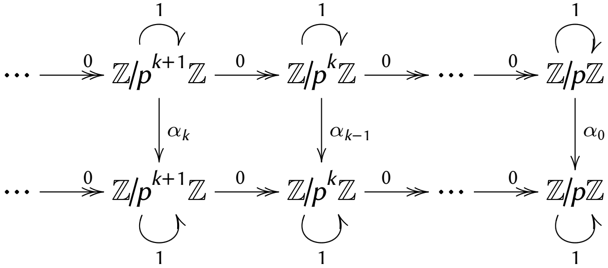

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Ultrastructure on $\mathbf{Mod}(\mathbf{FinSet}^{\mathbb{Z}})$

Ultrafilters and Ultracategories

An ultrafilter μ on a set S is a morphism $\mu : \wp(S) \to \mathbf{2}$.

Equivalently, $\mu \subseteq \wp(S)$ is a maximal (prime) filter.

μ is *cofiltered*: $U, V \in \mu$ implies $U \cap V \in \mu$.

An *ultracategory* $\mathcal{M}$ is a category $\mathcal{M}$ such that for any S, μ,

- There is a functor $\int(-)d\mu : \mathcal{M}^S \to \mathcal{M}$.
- These functors satisfy some further coherent data.

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Ultrastructure on Set

For $f\colon S \to \mathbf{Set}$, for $B \subseteq A \subseteq S$, there is a canonical projection

$$\prod_{s\in A} fs \to \prod_{s\in B} fs.$$

Given an ultrafilter μ on S, the ultraproduct is defined as follows,

$$\int f d\mu = \varinjlim_{A\in\mu} \prod_{s\in A} fs.$$

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Ultrastructure on Discrete Categories

Given a set X considered as a discrete category, an ultrastructure is the same as a *compact Hausdorff topology*,

$$\mathbf{UltSet} \cong \mathbf{Comp}.$$

Let $X \in \mathbf{Comp}$. Only consider $id : X \to X$ and μ on X:

- $\int \mathrm{id}\, d\mu$ is the *convergent point* under μ.
- $\int \mathrm{id}\, d\mu = x$ iff $\tau_x \subseteq \mu$.

For general $f : S \to X$, μ on S, consider the convergent point of $f_* \mu$,

$$U \in f_*(\mu) \Leftrightarrow f^{-1}(U) \in \mu.$$

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Łos Ultraproduct Theorem

Theorem (Łos Ultraproduct Theorem)

Given an S-indexed family of models $\{M_s\}_{s\in S}$, the ultraproduct $\int M_s d\mu$ for any ultrafilter μ on S is again a model.

Proof.

We need to verify that the following composite is coherent,

$$\mathcal{C} \xrightarrow{\{M_s\}_{s\in S}} \mathbf{Set}^S \xrightarrow{\int(-)d\mu} \mathbf{Set}$$

- $\{M_s\}_{s\in S}$ is coherent because each M_s is.
- $\int(-)d\mu : \mathbf{Set}^S \to \mathbf{Set}$ is coherent essentially because it is a filtered colimit, and that commutes with finite limits. □

In particular, computation is point-wise: $\int M_s d\mu(C) \cong \int M_s(C) d\mu$.

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Ultrastructure on $\mathbf{Mod}(\mathbf{FinSet}^{\mathbb{Z}})$

Consider an ultrafilter μ on S and the functor

$$\int (-) d\mu : \mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}})^S \to \mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}}).$$

The relevant data is a function $\int (-) d\mu : \widehat{\mathbb{Z}}^S \to \widehat{\mathbb{Z}}$.

For the p-adic component, the convergence of $\{\alpha_s\}_{s \in S} \in \mathbb{Z}_p^S$ is determined by the convergence of each $\{\alpha_{s,k}\}_{s \in S} \in (\mathbb{Z}/p^k\mathbb{Z})^S$,

$$\begin{array}{c} S \xrightarrow{\alpha_{s,k}} \mathbb{Z}/p^k\mathbb{Z}, \quad S \xrightarrow{\alpha_{s,k-1}} \mathbb{Z}/p^{k-1}\mathbb{Z}, \quad \cdots, \quad S \xrightarrow{\alpha_{s,1}} \mathbb{Z}/p\mathbb{Z} \\ \cdots \twoheadrightarrow \mathbb{Z}/p^k\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{k-1}\mathbb{Z} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \end{array}$$

This equips $\mathbb{Z}_p$ with the profinite topology as the following limit

$$\mathbb{Z}_p \cong \varprojlim \left(\cdots \twoheadrightarrow \mathbb{Z}/p^k\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{k-1}\mathbb{Z} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \right).$$

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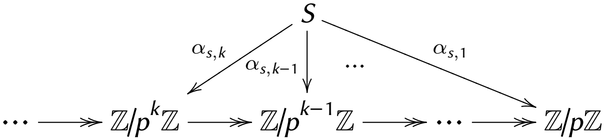

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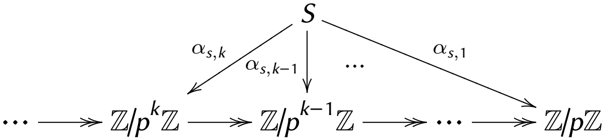

This equips $\mathbb{Z}_p$ with the profinite topology as the following limit

$$\mathbb{Z}_p \cong \varprojlim \left(\cdots \twoheadrightarrow \mathbb{Z}/p^k\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{k-1}\mathbb{Z} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \right).$$

Conclusion and Future Work

We have shown an equivalence $\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{Z}}) \simeq \widehat{\mathbb{Z}}$. This provides a model-theoretic account of the profinite topology on $\widehat{\mathbb{Z}}$.

[Ult] has proved an equivalence

$$\mathbf{Stone}_{\mathbf{FinSet}^{\mathbb{Z}}} \simeq \mathbf{Pro}^{wp}(\mathbf{FinSet}^{\mathbb{Z}}),$$

- Objects as $(X, \mathcal{O}_X)$: $X \in \mathbf{Stone}$, $\mathcal{O}_X \in \mathrm{Mod}_{\mathbf{FinSet}^{\mathbb{Z}}}(\mathrm{Sh}(X))$.
- Morphisms as (f, φ): $f : X \to Y$, $\varphi : f^*\mathcal{O}_Y \to \mathcal{O}_X$.

For $\mathbf{FinSet}^{\mathbb{Z}}$, we can show

$$\mathrm{Mod}_{\mathbf{FinSet}^{\mathbb{Z}}}(\mathrm{Sh}(X)) \simeq \mathrm{Bun}_{\widehat{\mathbb{Z}}}(X).$$

We can use this equivalence to classify profinite dynamical systems.

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In fact, the same argument is valid for an arbitrary group $\mathbb{G}$,

$$\mathrm{Mod}(\mathbf{FinSet}^{\mathbb{G}}) \simeq \widehat{\mathbb{G}},$$

where $\widehat{\mathbb{G}}$ is the profinite completion of $\mathbb{G}$.

This story is probably well-known in topos theory (cf. [Sheaves]).
Our argument provides a new site definition of $B\widehat{\mathbb{G}}$,

$$\mathrm{Sh}(\mathbf{FinSet}^{\mathbb{G}}) \simeq B\widehat{\mathbb{G}} \simeq \mathrm{Sh}(\mathbf{S}(\widehat{\mathbb{G}}), \mathbf{J}^{at}).$$

In particular, this shows each $B\widehat{\mathbb{G}}$ is coherent.

But explicit pretopos might simplfy model-theoretic Galois theory,

$$[\mathcal{C}, \mathbf{FinSet}^{\mathbb{G}}]^* \simeq \{\mathcal{C}\text{-models with continuous } \widehat{\mathbb{G}}\text{-action}\},$$

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