Rezk Completions

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Review: Rezk completion of a category

Goal: Rezk completion of a monoidal category

Future work: Rezk completion of structured categories

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One approach to make it precise is using univalent foundations and univalent categories.

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Lemma

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Definition

A category $\mathcal{C}$ is **univalent** if for any $x, y : \mathcal{C}$, the function $\mathsf{idtoiso}_{x,y}$ is an equivalence of types.

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Non-univalent category:

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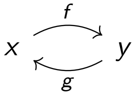

such that $f \cdot g = \mathsf{Id}x$ and $g \cdot f = \mathsf{Id}y$.

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2. Any functor $F : \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ univalent, factors *uniquely* via $\mathcal{H}$:

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle\mathcal{H}}\big\downarrow & \searrow^{F} & \\ \mathrm{RC}(\mathcal{C}) & \overset{}{\underset{\exists!}{\dashrightarrow}} & \mathcal{E} \end{array}$$

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1. *Equivalently:* $\mathcal{H} \cdot (-)$ *is adjoint equivalence of categories.*
2. *Equivalently:* $\mathcal{H} \cdot (-)$ *is weak equivalence of categories.*

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such that for any univalent monoidal category $\mathcal{E}$,

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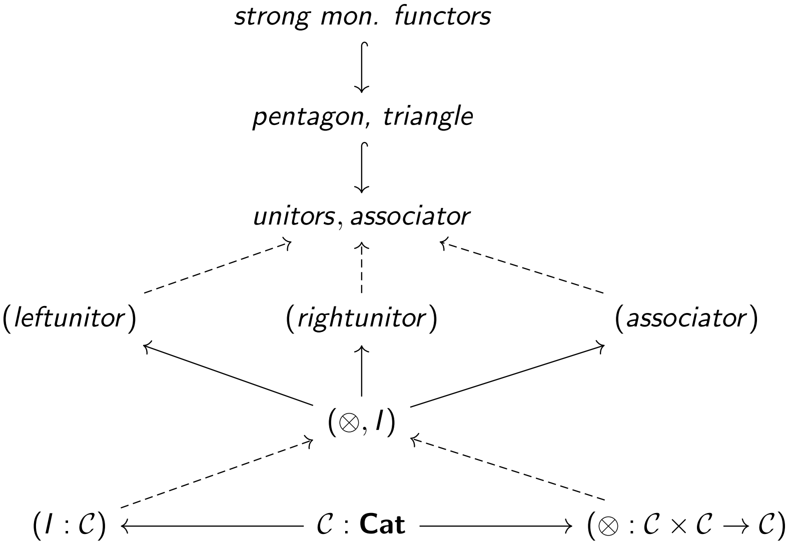
Monoidal Rezk completion: Approach



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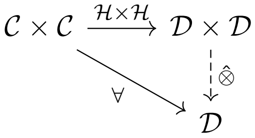

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3. $\mathcal{H}$ preserves the tensor strongly.

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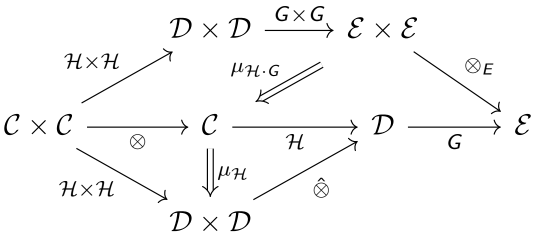

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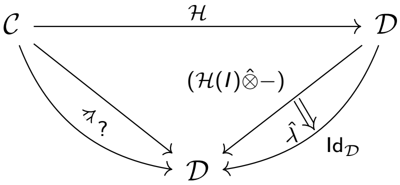

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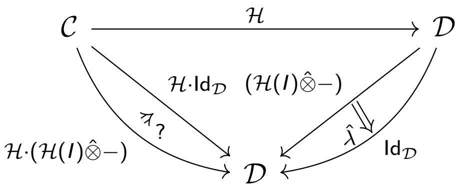

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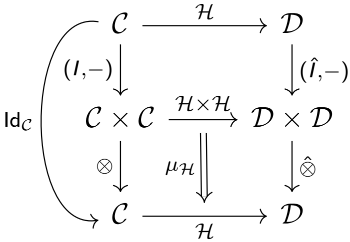

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4. Strong monoidal functors: Lift of natural iso is a natural iso.

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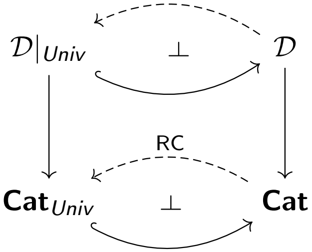
2. 

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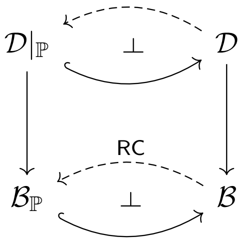

Goal: Study of lifting reflective sub-bicategories.

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Then $\mathcal{D} := \mathrm{Alg}(F)$ *admits Rezk-completions.*

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A signature for those functors:

$$F := [c \mid \mathrm{Id} \mid F + F \mid F \times F]$$

Final slide

THANK YOU!