Type Formers in Directed Type Theory

Benedikt Ahrens, Paige Randall North, **Niels van der Weide**

24 November, 2022

Higher dimensional type theory

- Martin-Löf's identity type gives types the structure of **higher groupoids**
- This led to the development of **homotopy type theory** (HoTT)
- **Synthetic algebraic topology**: done via HoTT
- **Directed type theory**: directed version of HoTT
- Directed topological spaces are used to study concurrency [1], and directed type theory is conjectured to model such spaces.

[1] Fajstrup, Lisbeth, et al. *Directed algebraic topology and concurrency*. Vol. 138. Berlin: Springer, 2016.

Directed type theory

Directed variants of type theory:

- An interpretation with directed definitional equality[2]
- A syntactical framework for directed type theory[3]
- An interpretation with directed identity types[4]

Interpreted in something like **categories**

[2] Licata, Daniel R., and Harper, Robert. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

[3] Nuyts, Andreas. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

[4] North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

Goal of this talk

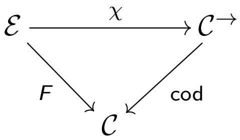
- Provide a setting in which one can interpret directed dependent type theory: **comprehension bicategory**
- Type formers in fibrations of bicategories: **hyperdoctrines**

Comprehension Categories

Type theory can be interpreted in **comprehension categories**.

Definition

A **comprehension category** is a *strictly* commuting triangle



where F is a Grothendieck fibration and where χ preserves cartesian cells.

Fibrations of Bicategories

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Global condition:
Given a substitution $s : \Gamma_1 \to \Gamma_2$ and type A in context Γ_2, we get a type $A[s]$ in context Γ_1.
This is *substitution on types*.

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Local condition:
Given a 2-cell $\tau : s_1 \Rightarrow s_2$ where $s_1, s_2 : \Gamma_1 \to \Gamma_2$, and a term $t : A[s_1]$, we get a term of type $A[s_2]$.
(think of 2-cells $\tau : s_1 \Rightarrow s_2$ as reductions from s_1 to s_2)

Comprehension Bicategories

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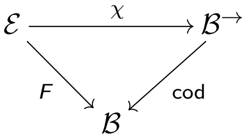

where χ preserves cartesian cells and where F is a global fibration and a local opfibration.

Example

We have the following comprehension bicategory

$$\begin{array}{ccc} \mathsf{OpFib} & \longrightarrow & \mathsf{Cat}^{\rightarrow} \\ & {\scriptstyle \mathsf{cod}}\searrow \quad \swarrow{\scriptstyle \mathsf{cod}} & \\ & \mathsf{Cat} & \end{array}$$

This can be generalized to arbitrary bicategories by using **internal Street (op)fibrations**.

Example

We have the following comprehension bicategory

$$\begin{array}{ccc} \mathsf{Cat} \,/\, \mathsf{CAT} & \xrightarrow{\quad\int\quad} & \mathsf{Cat}^{\rightarrow} \\ & {\scriptstyle \mathsf{dom}} \searrow \quad \swarrow {\scriptstyle \mathsf{cod}} & \\ & \mathsf{Cat} & \end{array}$$

Goal: type formers

- Comprehension bicategories allow us to interpret the judgmental structure of type theory
- However, type theory without types is rather boring
- Next goal: interpreting type formers

Fibers

Note: to formulate type formers, we look at **fibers**.
Suppose that we have a fibration $p : E \to B$.
Fiber bicategory E_b over $b : B$:

- Objects: $\overline{b}$ over b
- 1-cells: $\overline{f} : \overline{b} \to \overline{b'}$ over id
- 2-cells: $\overline{\tau} : \overline{f} \Rightarrow \overline{g}$ over id

Since we have a fibration, every $f : b_1 \to b_2$ gives rise to a functor $E_f : E_{b_2} \to E_{b_1}$.

Simplified Setting

We work in the following setting

- We have a fibration $p : E \to B$ of bicategories
- The fiber of every $b : B$ is a category.
- This means: all 2-cells τ_1, τ_2 in E that live over a θ in b are equal and every τ over an invertible θ is again invertible.

Example

$$
\begin{array}{c}
\mathsf{Cat} \,/\, \mathsf{Set} \\
\Big\downarrow{\scriptstyle \mathsf{dom}} \\
\mathsf{Cat}
\end{array}
$$

Here:

- Objects in Cat / Set are functors $F : C \to \mathsf{Set}$
- 1-cells from $F : C_1 \to \mathsf{Set}$ to $G : C_2 \to \mathsf{Set}$ consist of a functor $H : C_1 \to C_2$ and an invertible natural transformation from F to $G \circ H$.
- 2-cells from $H : C_1 \to C_2$ to $H' : C_1 \to C_2$ are natural transformations τ such that some diagram commutes (I won't give that diagram here)

Example

$$\begin{array}{c} \mathsf{FF} \\ {\scriptstyle \mathsf{cod}}\downarrow \\ \mathsf{Cat} \end{array}$$

Here:

- Objects in FF are fully faithful functors $F : C \to D$
- 1-cells from $F : C_1 \to D_2$ to $G : C_2 \to D_2$ consist of functors $H : C_1 \to C_2$ and $K : D_1 \to D_2$ and an invertible natural transformation from $K \circ F$ to $G \circ H$.
- 2-cells from $H_1 : C_1 \to C_2$ and $K_1 : D_1 \to D_2$ to $H_1 : C_1 \to C_2$ and $K_1 : D_1 \to D_2$ consist of natural transformations $\tau : H_1 \Rightarrow H_2$ and $\theta : K_1 \Rightarrow K_2$ making some diagram (that I don't give here) commute

Example of Fibers

Fibration	Fiber category over $\mathcal{C}$	Fiber functor
$\mathsf{FF} \xrightarrow{\text{cod}} \mathsf{Cat}$	Fully faithful functors into $\mathcal{C}$	Precomposition
$\mathsf{Cat} / \mathsf{Set} \xrightarrow{\text{dom}} \mathsf{Cat}$	$\mathcal{C} \to \mathsf{Set}$	Pullback

Simple type formers

Definition

A fibration supports **conjunction** if

- The fiber category over every $\mathcal{C}$ has products
- The fiber functor preserves products

Same for disjunction, implication, and negation.

Quantifiers as adjoints

Definition

A fibration $p : E \to B$ has existential types if for every $f : b_1 \to b_2$ the functor E_f has a left adjoint.

For dom : $\mathrm{Cat} \,/\, \mathrm{Set} \to \mathrm{Cat}$: we need a left adjoint for precomposition.

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This is left Kan extension!
But what about Beck-Chevalley?

Beck-Chevalley for left Kan extensions

Suppose that we have the following square

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C_3 & \xrightarrow[h]{} & C_4 \end{array}$$

Let lan_f denote the left Kan extension of f.
What can we say about $\mathsf{lan}_h \cdot k^* \Rightarrow g^* \cdot \mathsf{lan}_f$?

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Let lan_f denote the left Kan extension of f.
What can we say about $\mathsf{lan}_h \cdot k^* \Rightarrow g^* \cdot \mathsf{lan}_f$?

- If the above square is a comma square: it is invertible
- If the above square is a pullback: not much...

Conclusion

- This is very much work in progress.

Questions

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- This is very much work in progress.
- As such, there is no conclusion. There only are questions.
- What is the proper formulation of the Beck-Chevalley condition? How does this affect the syntax?
- How about identity types?