The internal language of sheaves

Applications to algebraic geometry

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Outline

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2. Kripke-Joyal Semantics
3. Modal Operators
4. Relation between formulas on $X_\Box$ and formulas on X + why it is unclear
5. Solution

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Internal language of a topos

A topos is a category that has all the properties of **Set** required for finitary reasoning. That is, we can do first order logic inside a topos.

Definition (Elementary topos)

An elementary topos is a category that

- has all finite limits
- is cartesian closed
- has a subobject classifier Ω

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Example

The category of sheaves over a topological space X, $\mathrm{Sh}(X)$, is a topos.

Internal language, context

To be able to do first order logic, we must be able to represent jugment i.e. : $\Gamma \vdash F$.

Context

- Objects $\mathcal{F}$ are our types
- Variables x of type $\mathcal{F}$ are interpreted as as the identity morphism $\mathrm{id} : \mathcal{F} \to \mathcal{F}$
- Terms σ of type $\mathcal{F}$ are interpreted as morphisms $\sigma = (\sigma_1, ..., \sigma_n) : \mathcal{E}_1 \times ... \times \mathcal{E}_n \to \mathcal{F}$.
- Formula are the terms $\mathcal{F} \to \Omega$

So $\Gamma \vdash F$, will be represented by $\varphi : \mathcal{E}_1 \times ... \times \mathcal{E}_n \to \Omega$.

Internal language

Moreover, we need to build constructors corresponding to $\bot$, $\top$, $\wedge$, $\vee$, $\Rightarrow$, $\forall$ and $\exists$.

Example (Connectives)

For any formulas $\varphi : \mathcal{F} \to \Omega$ and $\psi : \mathcal{F} \to \Omega$ we want to build a new formula

$$\varphi \wedge \psi : \mathcal{F} \xrightarrow{\langle \varphi, \psi \rangle} \Omega \times \Omega \xrightarrow{\wedge_\Omega} \Omega$$
$$\varphi \vee \psi : \mathcal{F} \xrightarrow{\langle \varphi, \psi \rangle} \Omega \times \Omega \xrightarrow{\vee_\Omega} \Omega$$
$$\varphi \Rightarrow \psi : \mathcal{F} \xrightarrow{\langle \varphi, \psi \rangle} \Omega \times \Omega \xrightarrow{\Rightarrow_\Omega} \Omega$$

Proposition

We can also define $\top, \bot, \exists, \forall$

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Kripke-Joyal semantics

Property transfer idea

Properties in algebraic geometry

K-J semantics ↓ ↑ K-J semantics

Properties expressed in internal language

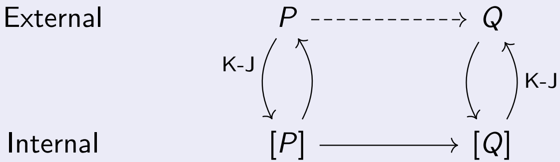
Properties we want to prove
Complicated objects

Translated properties
Simpler objects

Internal proofs 101

General scheme of internal proofs

Suppose we have a properties P and Q and we want to give a proof of $P \to Q$. Then we can translate P and Q to their corresponding internal statements $[P]$ and $[Q]$ and give a proof of $[P] \to [Q]$ instead.



Kripke-Joyal semantics

Notation

Given a formula $\varphi : \mathcal{F} \to \Omega$ *and an open set* $U \subseteq X$*, we have* $\varphi_U : \mathcal{F}(U) \to \Omega(U)$*. Given* $\alpha \in \mathcal{F}(U)$*, we will write* $\varphi(\alpha)$ *instead* $\varphi_U(\alpha)$

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Definition (Kripke-Joyal semantics)

Let X be a topological space, $U \subseteq X$ and φ be some formula $\varphi : \mathcal{F} \to \Omega$ in the internal language of $\mathrm{Sh}(X)$ then.

$$U \vDash \varphi(\alpha) \text{ for } \alpha \in \mathcal{F}(U) \quad \text{iff} \quad \alpha \in \{x|\varphi(x)\}(U)$$

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Example (Recursive unwinding)

- $U \vDash (\varphi \wedge \psi)(\alpha)$ iff $U \vDash \varphi(\alpha)$ and $U \vDash \psi(\alpha)$.
- $U \vDash (\varphi \Rightarrow \psi)(\alpha)$ iff for all open $V \subseteq U$ $V \vDash \varphi(\alpha|_V)$ implies $V \vDash \psi(\alpha|_V)$.

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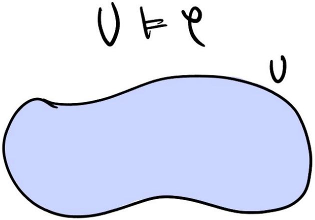
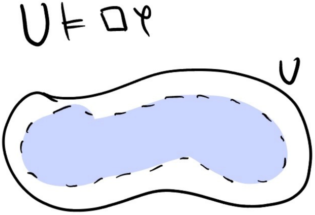
Modalities

Sometimes having a property holding globally is to strict, we also want to be able to describe "local" properties.

We want some operator $\Box : \Omega \to \Omega$ that does this:





Modal operators

Definition (Modal operator)

A modal operator is a map $\Box : \Omega \to \Omega$ such that

1. $\varphi \implies \Box\varphi$
2. $\Box\Box\varphi \implies \Box\varphi$
3. $\Box(\varphi \wedge \psi) \iff \Box\varphi \wedge \Box\psi$

Modal operators

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2. $\square\square\varphi \implies \square\varphi$
3. $\square(\varphi \wedge \psi) \iff \square\varphi \wedge \square\psi$

Examples (Some general modal operators)

For a fixed formula in the internal language α, the following are modal operators

1. $\square\varphi := (\alpha \Rightarrow \varphi)$
2. $\square\varphi := ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$
3. $\square\varphi := \neg\neg\varphi$

Specific modal operators and their translations

Since the open set $U \subseteq X$ is an element of $\Omega(X)$ we have that $V \vDash U \iff V \subseteq U$.

Definition ($!x$)

Because $x \notin V \iff V \vDash \text{int}(X \setminus \{x\})$ we let $!x$ denote $\text{int}(X \setminus \{x\})$.

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Proposition (Instances of modal operators)

For $x \in X$ we have that for all open $V \subseteq X$

- $V \vDash \neg\neg\varphi \iff \exists W \subseteq V$ *which is open and dense such that* $W \vDash \varphi$
- $V \vDash ((\varphi \Rightarrow !x) \Rightarrow !x) \iff x \notin V$ *or* $\exists W \subseteq V$ *with* $x \in W$ *and* $W \vDash \varphi$

$X_\square$, the associated map j

Sometimes studying these local properties directly on X can be hard, thus we will define a subspace, $X_\square$, which validates the properties on which they are easier to study.

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Definition (Associated map to a modal operator)

A modal operator induces a map on the elements of $\Omega(X)$ (i.e. the open subsets of X)

$$\begin{aligned} j : \Omega(X) &\longrightarrow \Omega(X) \\ U &\longmapsto \bigcup\{V \subseteq X \mid V \text{ open}, \ V \vDash \square U\} \end{aligned}$$

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Proposition (Properties of j)

In particular we have that for any open $U, V \subseteq X$

1. $U \subseteq j(U)$
2. $j(j(U)) \subseteq j(U)$
3. $j(U \cap V) = j(U) \cap J(V)$

$X_\square$ as an induced subspace

Definition (Subspace associated to a modal operator)

j defines a subspace of X, denoted by $X_\square$. It has a frame of opens $\mathcal{T}(X_\square) := \{U \in \Omega(X) \mid j(U) = U\}$

$X_\square$ as an induced subspace

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Modal operator	associated map	$X_\square$
$\square\varphi \equiv (U \Rightarrow \varphi)$	$j(V) = \text{int}(U^c \cup V)$	U
$\square\varphi \equiv \neg\neg\varphi$	$j(V) = \text{int}(\text{cl}(V))$	smallest dense sublocale of X
$\square\varphi \equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$j(V) = \begin{cases} X \text{ if } x \in V \\ X \setminus \text{cl}(\{x\}) \text{ if } x \notin V \end{cases}$	$\{x\}$

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Relation between formulas in $\mathrm{Sh}(X_\Box)$ and those in $\mathrm{Sh}(X)$

Theorem ($\Box$'d formula relation)

Let X be a topological space and $\Box$ be a modal operator on $Sh(X)$ then for any formula φ

$$X \vDash_{Sh(X)} \varphi^{\Box} \iff X_\Box \vDash_{Sh(X_\Box)} \varphi$$

where all parameters on the right side are pulled back to $X_\Box$ along $X_\Box \hookrightarrow X$.

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Proposition ($\Box$'d formula relation for geometric formulas)

For geometric formulas and geometric implications φ we have

$$\Box\varphi \iff \varphi^{\Box}$$

$$X \vDash_{Sh(X)} \Box\varphi \iff X_\Box \vDash_{Sh(X_\Box)} \varphi$$

Algebraic geometry

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For geometric formulas and geometric implications φ we have

$$X \vDash_{Sh(X)} \Box\varphi \iff X_\Box \vDash_{Sh(X_\Box)} \varphi$$

Example (Properties on the space vs at stalks)

Let $x \in X$ and $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$, then by the theorem

$$X \vDash \varphi^\Box \iff \varphi \text{ holds at the stalk at x}$$

From the proposition we get that a geometric implication holds if and only if it holds at every stalk.

Problem

Theorem

Let X be a topological space and $\Box$ be a modal operator on $Sh(X)$ then for any formula φ

$$X \vDash_{Sh(X)} \varphi^{\Box} \iff X_{\Box} \vDash_{Sh(X_{\Box})} \varphi$$

where all parameters on the right side are pulled back to $X_{\Box}$ along $X_{\Box} \hookrightarrow X$.

Question

Since the formula φ in $\mathrm{Sh}(X)$ can have a context that is not a $\Box$-sheaf as well as domains of quantification's that are not $\Box$-sheaves, does the equivalence

$$X \vDash_{\mathrm{Sh}(X)} \varphi^{\Box} \iff X_{\Box} \vDash_{\mathrm{Sh}(X_{\Box})} \varphi$$

make sense?

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Solution

Outline of solution

1. We are going to define a construction $+$ such that for all sheaves $\mathcal{F}$, $\mathcal{F}^{++}$ is a $\Box$-sheaf.
2. We are going to define an associated translation on formulas such that for a formula φ, $\varphi^{\Box} \Leftrightarrow (\varphi^{\Box})^{++}$
3. This will enable us to prove

$$X \vDash_{\mathsf{Sh}(X)} (\varphi^{\Box})^{++} \iff X_{\Box} \vDash_{\mathsf{Sh}(X_{\Box})} \varphi^{++}$$

4. And we can conclude

$$X \vDash_{\mathsf{Sh}(X)} \varphi^{\Box} \iff X_{\Box} \vDash_{\mathsf{Sh}(X_{\Box})} \varphi$$

Sheaves on $X_\Box$

What do sheaves on $X_\Box$ look like?

Definition (□-sheaf)

A □-sheaf is a sheaf in $\mathrm{Sh}(X)$ that "looks" like a sheaf on $X_\Box$

Proposition

There is an equivalence of categories $Sh(X_\Box) \simeq Sh_\Box(Sh(X))$, where $Sh_\Box(Sh(X))$ is the category of □-sheaves in $Sh(X)$. It is induced by the following map

$$i : \mathcal{O}(X) \to \mathcal{T}(X_\Box)$$

$$U \mapsto j(U)$$

How to construct $\mathcal{F}^{++}$

Definition (Johnstone construction/plus construction)

Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$ and $f : \mathcal{F} \to \mathcal{G}$, then

- $\mathcal{F}^+ = \{S \subseteq \mathcal{F} \; \Box(\ulcorner S \text{ is singleton} \urcorner)\}/\sim$, where $S \sim T :\Leftrightarrow \Box(S = T)$
- $f^+ : \mathcal{F}^+ \to \mathcal{G}^+, \;\; [S] \mapsto [\{f(x) | x \in S\}]$
- $\gamma : \mathcal{F} \to \mathcal{F}^+$, $x \mapsto [\{x\}]$

Proposition

For a sheaf $\mathcal{F}$, $\mathcal{F}^{++}$ is a $\Box$-sheaf.

How to construct φ^{++}

Proposition

Let $\Box$ be a modal operator, φ be some formula, Then $\varphi^{\Box} \Leftrightarrow ((\varphi^{\Box})^{+})^{+}$ intuitionistically.

Definition (+-translation)

- We replace $\overline{x} \in \mathcal{F}$ by $\overline{x} \in \mathcal{F}^{+}$.
- Terms $x_1, ..., x_n$ and $f(\overline{y})$ are replaced by $\gamma(x_1), ..., \gamma(x_n)$ and $f^{+}(\overline{y})$.
- φ^{+} of a formula $\varphi : \mathcal{F} \to \Omega$ is attained by replacing all free variables with their γ-images and morphisms and domains of quantifications with their $+$-constructions, e.g. $(\forall x : \mathcal{F}\ f(x) = g(x))^{+} := \forall x : \mathcal{F}^{+}\ f^{+}(x) = g^{+}(x)$.

Conclusion

Theorem

Let X be a topological space and $\Box$ be a modal operator on $Sh(X)$ then for any formula φ

$$X \vDash_{Sh(X)} (\varphi^{\Box})^{++} \iff X_{\Box} \vDash_{Sh(X_{\Box})} \varphi^{++}$$

Now when proving the theorem we can pretend that the context and that all domains of quantification are $\Box$-sheaves.

Theorem

Let X be a topological space and $\Box$ be a modal operator on $Sh(X)$ then for any formula φ

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