Monotone Type Theory: The Simplex Category as a Classifying Category

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The Simplex Category

- Δ is the simplex category
- Objects are nonempty, finite ordinals

$$[n] = \{0, \dots, n\} = n + 1$$

- Morphisms $f : [n] \to [m]$ are monotone functions

$$j \leq k \text{ implies } f(j) \leq f(k)$$

Standard Simplices

- The standard simplex functor embeds Δ into Top

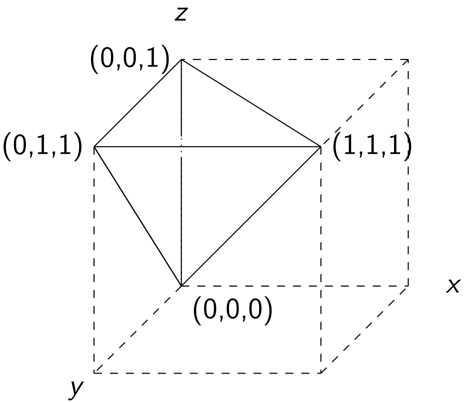
$$\begin{array}{rcl} \Delta & \hookrightarrow & \mathrm{Top} \\ [n] & \mapsto & \Delta^n \end{array}$$

- The standard n-simplex is the subspace

$$\Delta^n = \{(t_1, \ldots, t_n) \mid t_1 \leq \ldots \leq t_n\} \subseteq [0,1]^n \subseteq \mathbb{R}^n$$

Example

$$\Delta^3 = \{(t_1, t_2, t_3) \mid t_1 \leq t_2 \leq t_3\} \subseteq [0,1]^3$$



Type Theoretic Presentation of Δ

1. Develop monotone type theory $\mathbb{T}_m$ with $\mathbb{C}_{\mathbb{T}_m} \cong \Delta$
2. Present sound and strongly complete semantics for $\mathbb{T}_m$

Overview

1. Background
2. Syntax
3. Semantics
4. Future Research

Cartesian Type Theory

- Cartesian type theory $\mathbb{T}_c$ has one type $\mathbb{I}$ and two constants $0 : \mathbb{I}$, $1 : \mathbb{I}$
- $[x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}] \vdash t : \mathbb{I}$ iff $t \in \{0, x_1, \ldots, x_n, 1\}$
- A model in a category $\mathbb{C}$ is a bipointed object in $\mathbb{C}$

$$[\![\mathbb{I}]\!]^0 \underset{[\![0]\!]}{\overset{[\![1]\!]}{\rightrightarrows}} [\![\mathbb{I}]\!]$$

Classifying Category

- $\mathbb{C}_{\mathbb{T}_c}$ is the classifying category of $\mathbb{T}_c$
- Objects are contexts $[x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}]$
- Morphisms are context morphisms

$$\langle t_1, \ldots, t_m \rangle : [x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}] \to [y_1 : \mathbb{I}, \ldots, y_m : \mathbb{I}]$$

where $t_j \in \{0, x_1, \ldots, x_n, 1\}$

- $\mathbb{C}_{\mathbb{T}_c} \cong \square =$ Cartesian cube category, objects are $\mathtt{I}^n$

Martin-Löf Type Theory

- Constructive foundation of mathematics
- Dependently typed programming language
- Assuming univalence as an axiom breaks the Curry-Howard isomorphism: No longer a programming language

Cartesian Cubical Type Theory

- There is a model of Martin-Löf type theory in cSet $= [\square^{op}, \mathrm{Set}]$
- Allows a computational interpretation of univalence
- Pull features of cSet model back into the syntax
- Augment Martin-Löf type theory with Cartesian type theory to talk about $\square \hookrightarrow \mathrm{cSet}$

Simplicial Type Theory

- Desirable to have analogous extension of type theory based on $\mathrm{sSet} = [\Delta^{\mathrm{op}}, \mathrm{Set}]$
- Simplicial methods more common than cubical methods in homotopy theory and higher category theory
- First step is to give a type theoretic presentation of Δ

Embedding Δ into syntax

$$
\begin{aligned}
\Delta &\hookrightarrow \mathbb{C}_{\mathbb{T}_c} \\
[n] &\mapsto [x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}] \\
f &\mapsto \langle t_1, \ldots, t_m \rangle
\end{aligned}
$$

Monotone Context Morphisms

$$f : [n] \to [m] \text{ is mapped to}$$

$$\langle t_1, \ldots, t_m \rangle : [x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}] \to [y_1 : \mathbb{I}, \ldots, y_m : \mathbb{I}]$$

where

$$t_1 \leq \ldots \leq t_m$$

according to the linear order

$$0 \leq x_1 \leq \ldots \leq x_n \leq 1$$

Modifying Cartesian Type Theory

1. Introduce the binary predicate symbol $\leq : \mathbb{I}, \mathbb{I}$

$$[x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}] \vdash 0 \leq x_1 \leq \ldots \leq x_n \leq 1 : \mathbb{I}$$

2. Take away the structural rule of exchange

3. Generate only monotone context morphisms

$$\frac{\Gamma \vdash \langle \tau, t \rangle \Rightarrow [\Theta, y : \mathbb{I}] \qquad \Gamma \vdash t \leq u : \mathbb{I}}{\Gamma \vdash \langle \tau, t, u \rangle \Rightarrow [\Theta, y : \mathbb{I}, y' : \mathbb{I}]}$$

Result

- $\mathbb{C}_{\mathbb{T}_m}$ is the classifying category of $\mathbb{T}_m$
- Objects are contexts
- Morphisms are *monotone* context morphisms
- $\mathbb{C}_{\mathbb{T}_m} \cong \Delta$

Intervals in a Topos

- An internal interval $\mathtt{I}$ in a topos $\mathcal{E}$ is a linear order with top and bottom elements

$$[\![\mathbb{I}]\!]^0 \underset{[\![0]\!]}{\overset{[\![1]\!]}{\rightrightarrows}} [\![\mathbb{I}]\!] = \mathtt{I} \qquad [\![\leq]\!] \hookrightarrow [\![\mathbb{I}]\!]^2 = \mathtt{I}^2$$

- An internal n-simplex is a subobject

$$\Delta^n_{\mathtt{I}} = \{(x_1, \ldots, x_n) \mid x_1 \leq \ldots \leq x_n\} \hookrightarrow \mathtt{I}^n$$

Modelling Monotone Type Theory

- $\leq : \mathbb{I}, \mathbb{I}$ interpreted as $[\![\leq]\!] \hookrightarrow [\![\mathbb{I}]\!]^2 = \mathtt{I}^2$
- The order of $\Gamma = [x_1 : \mathbb{I}, \ldots, x_n : \mathbb{I}]$ is reflected in

$$\Delta^n_{\mathtt{I}} = \{(x_1, \ldots, x_n) \mid x_1 \leq \ldots \leq x_n\} \hookrightarrow [\![\Gamma]\!] = [\![\mathbb{I}]\!]^n = \mathtt{I}^n$$

Modelling Monotone Type Theory

- The monotonicity of context morphisms is reflected in

$$\begin{array}{ccc}
\Delta^n_{\mathbb{I}} & \dashrightarrow & \Delta^m_{\mathbb{I}} \\
{\scriptstyle \iota_n}\big\downarrow & & \big\downarrow \\
[\![\Gamma]\!] & \xrightarrow[{[\![\langle t_1,\ldots,t_m\rangle]\!]}]{} & [\![\Theta]\!]
\end{array}$$

Properties of Semantics

- Sound because of properties of internal intervals
- Strongly complete because of model in sSet given by the generic interval

$$\Delta(-,[1]) : \Delta^{\mathsf{op}} \to \mathsf{Set}$$

Future Research

- Develop a type theoretic presentation of cofibrations in sSet
- A 2-categorical perspective on Δ

Thank you!